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C. 1

SOME EQUILIBRIUM AND FREE VIBRATION PROBLEMS ASSOCIATED WITH CENTRIFUGALLY STABILIZED DISK AND SHELL STRUCTURES

by Walter Eversman

Prepared by
WICHITA STATE UNIVERSITY
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ERRATA

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2nd page on notation:

n is also used for the number of nodal diameters in the case of shell vibrations

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Equation (5.3.1) should read

$$D\left\{\phi_{rr} + \frac{\phi_r}{r} - \frac{\phi}{r^2}\right\} - \frac{\phi\psi}{r} + \frac{\psi}{R} = 0$$

Equation (5.3.2) should read

$$\frac{1}{Et}\left\{\psi_{rr} + \frac{\psi_r}{r} - \frac{\psi}{r^2}\right\} - \frac{\phi}{R} + \frac{1}{2}\frac{\phi^2}{r} = -\frac{(3+\nu)\rho\omega^2 r}{Et}$$

Page 139:

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Page 142:

Equation (5.9.15) should read

$$M_{\theta\theta} = -\frac{(3+\nu)\rho\omega^2 b^3 \ell^2}{R}\left(\frac{1}{\lambda^2}\right)\left[\nu g_x + g/x\right]$$

Equation (5.9.15) should read

$$\sigma_{\theta_B} = \pm\frac{6(3+\nu)\rho\omega^2 b^2}{t}\frac{1}{\sqrt{12(1-\nu^2)}}\left(\frac{1}{\lambda^2}\right)\left[\nu g_x + g/x\right]$$

Page 164:

Equation (6.2.32) should read

$$\cdots + \frac{1-\nu}{2}v_x - \frac{n\gamma(1+\nu)}{2}g(x)w_x - \cdots$$

Equation (6.2.33) should read

$$\cdots + \frac{\lambda^4\gamma}{x}\left[n^2\frac{h(x)}{x}w - \frac{n}{2(1+\nu)}g(x)S_{r\theta}\right] + \cdots$$

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$$\lambda = \left[\frac{Etb^4}{DR^2}\right]^{1/4} \approx 2\sqrt[4]{3(1-\nu^2)}\sqrt{R/t}$$

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SOME EQUILIBRIUM AND FREE VIBRATION PROBLEMS ASSOCIATED WITH CENTRIFUGALLY STABILIZED DISK AND SHELL STRUCTURES

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ABSTRACT

The basic governing equations for the study of problems of equilibrium and free vibration of shallow, isotropic, spherical shells spinning about their polar axis are derived from a variational principle. These equations are specialized to the case of a spinning flat membrane disk and the problem of transverse vibrations is solved for the case of a fully clamped hub and the case of a loosely clamped hub on an annular disk. The method of approach for intermediate hub configurations is also discussed.

The vibration analysis of shallow shells is approached and it is noted that the spinning shell equilibrium problem must be solved first to provide the necessary stress and displacement distributions for the vibration equations of motion. The equilibrium problem is treated by linear theory for a freely spinning shell and by both linear and nonlinear theory for the spinning shell with a fully clamped hub. It is found that bending effects are important, particularly for shells with a small

ratio of rise to thickness, and that a continuous transition from the shell solution to the solution for a spinning flat disk requires that bending be included. The nonlinear effect of finite rotations of shell elements is found to be strong, and the effect increases as the ratio of shell rise to shell thickness increases. It is concluded that a nonlinear theory which includes both membrane and bending effects should be employed for the general analysis of the equilibrium configuration.

The equations of motion for small transverse vibrations about the equilibrium configuration are formulated and reduced to a form suitable for numerical solution by an extension of a known method.

TABLE OF CONTENTS

NOTATION

B_{rr}, $B_{\theta\theta}$	bending stress "resultant", $B_{rr} = t\sigma_{r_B}$, etc.
C_n	coefficients in recurrence relations
D	bending stiffness, $D = \dfrac{Et^3}{12(1-\nu^2)}$
E	Young's modulus
M_{rr}, $M_{\theta\theta}$, $M_{r\theta}$	moment resultants
R	radius of curvature of shell
N_{rr}, $N_{\theta\theta}$	direct stress resultants
U, V, W	radial dependence of displacement components
a	hub radius in case of fully clamped disk or shell, inner annulus radius in case of loosely or partially clamped disk
b	outer radius of disk or shell
c	hub radius in case of loosely or partially clamped disk
e_r, e_θ	mid-surface strains
g	radius of fully clamped region in case of partially clamped disk
$g_{r\theta}$	midsurface shear strain
h	shell rise
i	$\sqrt{-1}$

l	a defined parameter in linear shallow shell theory, $l^2 = \frac{Rt}{\sqrt{12(1-\nu^2)}}$
m	mass per unit volume
n	number of nodal circles
p	natural frequency of vibration
r	radial co-ordinate
s	number of nodal diameters
t	time. Also shell or disk thickness
u, v, w	components of displacement
x	non-dimensional radius
z	axial co-ordinate
α	defined parameter, Eq. (3.4.2) for fully clamped disk, Eq. (4.4.2) for loosely clamped disk
$\alpha_n, \beta_n, \gamma_n, \delta_n$	coefficients in recurrence relation
γ	defined parameter, $\gamma = \frac{3+\nu}{Et} \rho \omega^2 R^2$
δ	defined parameter, Eq. (3.3.5) for fully clamped disk, Eq. (4.2.7) for loosely clamped disk. Also, $\delta = 1 - \sqrt{1-2\gamma}$ in nonlinear shell analysis.
ϵ	hub to disk radius ratio in case of loosely clamped hub
ζ	a non-dimensional independent variable
θ	polar co-ordinate angle

$\kappa_r, \kappa_\theta, \kappa_{r\theta}$	midsurface curvature changes
$\bar{\lambda}$	parameter defined by $\bar{\lambda} = \frac{1}{\sqrt{tED}}$
λ	defined parameter. An exact definition is given by $1/\lambda^4 = \frac{DR^2}{b^4Et}$ and an approximate definition by Eq. (5.4.11). Also used as unknown parameter in separation of variables.
μ	frequency parameter, defined by Eq. (4.3.4)
μ_1, μ_2, μ_3	parameters defined by Eqs. (3.2.11), (3.2.12) and (3.2.13)
ν	Poissons ratio
ρ	non-dimensional radius $\rho = \frac{r}{b}$. Also mass per unit surface area
ρ_o	used interchangably with m, mass per unit volume
σ_r, σ_θ	stresses
$\tau_{r\theta}$	shear stress
Φ	displacement variable
ψ	stress function
ω	spin angular velocity

CHAPTER I

INTRODUCTION

The purpose of this investigation is to formulate the equations of static and dynamic equilibrium for shallow spherical shells which are spinning about their polar axis, and to obtain solutions for the static stress distributions and free vibration natural frequencies and mode shapes for several cases of shell curvature and hub configuration. The shells considered in this analysis are assumed to conform to the usual restrictions of shallow shell theory and in addition are assumed to be isotropic. The static stress analysis is formulated allowing for finite rotations, but solutions are obtained both with this assumption and with the assumption of infinitesimal rotations. The free vibration problem is formulated as one of small perturbations about the static equilibrium configuration. Various aspects of this general problem area have been treated in the literature, however the unified formulation of the theory and the solutions for the special cases presented here are new.

The necessity for the design of efficient, lightweight structures for space applications has led to the consideration of centrifugally stabilized shells. In this type of structure advantage is taken of the fact that the static stress distribution in a shell spinning about its polar axis essentially generates a pre-stressed configuration which has a substantial resistence to deformation from the equilibrium shape even though the material has practically no inherent structural rigidity. In this way a shallow shell, such as a radar dish, can be constructed of a flexible, light-weight synthetic material and stabilized in shape by spinning it about its axis of symmetry.

Various applications of centrifugally stabilized shell structures have been proposed [1,2,3,4]. Included among these are collectors for solar furnaces, radio and radar antennas, solar sails, and drag modulation devices for re-entry from space. All these proposed uses have in common the necessity to provide a precise shell or disk geometry in the presence of known or random disturbances with a minimum of structural weight.

Initial interest in the solution of the vibration problems of spinning disks, the limiting case of a spinning shallow spherical shell, arose in connection with the design of high speed rotating machine elements such as saw blades and turbine disks. In general, a complete analysis of these problems would require the consideration of the bending stiffness of the disk material and the in-plane membrane stresses due to centrifugal body forces. Early investigators found that the solution of the general case posed considerable mathematical complexity and elected to study separately the cases in which rotational effects were negligible and the cases in which bending stiffness effects were overshadowed by rotational effects. Analysis of the situations in which bending stiffness is predominant led to a mathematical formulation for plate vibrations which was studied initially by Kirchoff and which is presently developed to a high degree.

The case in which rotational effects are of primary importance led directly to the study of spinning membrane disks. Lamb and Southwell [5,6] discuss the problem of small transverse vibrations of a spinning circular membrane which has no hole at the center and has a free outer edge. This problem is shown to reduce to that of obtaining solutions to the hypergeometric equation which are finite at two singularities. The solutions are in the form of Jacobi polynomials. Southwell [6] extends the analysis to the study of the transverse vibrations of a spinning membrane whose transverse deflections are constrained to be zero in an interval $r < a$ and whose outer edge is free. The central clamping is such that radial displacements are unrestrained. The solution to this problem necessitates finding hypergeometric functions which vanish at $r = a$ and are finite at the outer edge. Simmonds [7] considers the same problem and points out that Southwell's paper contains an error since it overlooks the fact that the hypergeometric equation cannot have two independent solutions involving only terms containing the hypergeometric series ${}_2F_1(\alpha, \beta, \gamma, x)$ if the parameter γ assumes integral values. Johnson [8] has also published solutions to the same problem as part of his investigations of the vibrations of shallow elastic membrane shells. His results reflect the same error as Southwell's for non-axially symmetric vibrations. Lamb and Southwell [5] use their membrane analysis results, Kirchoff's non-rotating results and Rayleigh's principle to approximate from below and above the natural frequencies of free vibrations for a spinning disk including both bending stiffness and rotational effects.

The case when the central clamping is such that radial displacement is completely or partially restrained is more difficult to analyze. The restraint on radial displacement gives rise to a more complex in-plane membrane stress distribution. The resulting differential equation governing the radial dependence of the vibration mode shapes is of the Fuchsian type with four regular singular points. In principal, this type of differential equation can be reduced to a form studied by Heun [9]. However, as discussed in the present analysis, it is more convenient not to make the reduction in this case. It is found that solutions for the radial mode shapes appear as power series whose coefficients depend on a four term recurrence relation. Hence, the problem with central clamping is correspondingly more difficult in the computational sense than is the problem without central clamping. Solutions for the natural frequencies of vibration for flat spinning membrane disks with several hub configurations constitute a portion of the present investigation.

Bulkeley and Savage [10] have studied the centrally clamped membrane for the case of axisymmetric vibrations. Their results are for various degrees of partial clamping up to and including the fully clamped configuration. Simmonds [11] has studied the fully clamped case for axisymmetric vibrations. Both investigations find that while the general vibrations are governed by an equation of motion which reduces to Heun's equation, the axisymmetric case reduces to the hypergeometric differential equation.

The most difficult boundary value problem for the spinning membrane occurs if it is annular and hubless. This case was studied by Eversman [12]. It is found in this case that reduction of the radial mode shape differential equation to Heun's equation is convenient. In this form, two of the singularities of Heun's equation correspond to the inner and outer edges of the annulus and the boundary value problem is that of finding solutions which are finite at two singular points. Eversman also found that the axisymmetric vibrations of the annular membrane reduced to the hypergeometric equation and that the case of one nodal diameter reduced to a form of Heun's equation which admits a polynomial solution. This was shown to correspond to a rigid body precession mode.

Interest in static and dynamic problems of spinning shells appears to be relatively recent. Explicit attention to the equilibrium stress distribution in a shallow spherical which is spinning about its polar axis was given minor attention by Reissner [13] in his important paper which presents solution techniques for the equations of shallow spherical shell theory in the case of axisymmetric loading. He gives a particular solution of the governing equations in the case of the inertia load on a rotating shell. No attempt is made to obtain a complete solution for various boundary conditions. The shell equations used by Reissner are based on the assumption of small displacements and rotations, a condition which might not be met if the shell is spinning rapidly or the stiffness of the shell material is low. A subsequent paper by Reissner [14] based on nonlinear shell and membrane theories of Marguerre [15], H. Reissner [16], and Bromberg and Stoker [17] allows for the possibility of finite rotations. A numerical integration scheme for nonlinear two-point boundary value problems for shells of revolution has been presented by Archer [18]. As a limiting case of the nonlinear shallow shell theory Simmonds [19] obtains solutions for the transverse displacement of a normally loaded spinning membrane disk.

Cohen [20,21] has solved the problem of the static equilibrium configuration of a spinning paraboloidal dish. His theory is based on a different set of governing linear equations than is Reissner's and has the potential shortcoming that all boundary conditions cannot be satisfied simultaneously.

The first investigator who specifically treated the statics and dynamics of spinning shallow membrane shells appears to have been Johnson [8]. His study is based on Reissner's nonlinear shallow shell theory for the membrane case. The vibration problem is considered to be one of small perturbations about the equilibrium state assumed by the spinning membrane shell. He finds that the equilibrium state is easily calculated and is one in which there is no meridional stress, independent of the curvature of the shell. This observation is inconsistent since it is known in the limiting case of a shallow shell which is a flat disk that the radial stress does not vanish. A consistent theory would show a continuous transition from the stress distribution of a shallow shell to that of a flat disk. This continuous transition is accounted for at the expense of a more complicated theory if bending effects are included. A second inconsistency in the membrane theory is the impossibility of satisfying certain boundary conditions, even if the stress distribution is primarily of a membrane type. As pointed out by Johnson, this difficulty is probably not serious since a boundary layer analysis [14] would show that bending effects are important only in the immediate vicinity of the edge for shell geometries for which membrane theory is nearly correct.

The general equations for equilibrium of rotationally symmetric spinning membrane shells were formulated from a variational principal by Simmonds [1]. He obtained the equations in a form such that the problem of determining the appropriate initial shell shape in the non-spinning state in order to obtain a desired shell configuration in the spinning state could be solved. As computational examples he cites the cases of the flat disk, sphere, and paraboloid with conical covering.

Other than Johnson's [8] investigations of the vibrations of shallow membrane shells, there have apparently been no published investigations of the free vibration characteristics of spinning shallow spherical shells. There have been numerous studies dealing with the free vibrations of stationary shallow spherical shells. Reissner [22,23] and Johnson and Reissner [24] neglect the effect of longitudinal inertia on the predominantly transverse vibrations of shallow shells. When this is done, it is found, just as in the static case, that the dynamic equations can be reduced to two simultaneous equations for the displacement and a stress function. Kalnins and Naghdi [25] extend the idea to three different degrees of neglect of the effect of longitudinal inertia. One of their forms of the equations corresponds to those used by Johnson and Reissner.

The case of vibrations of the spinning shell is significantly more complicated than the equivalent problem for the non-spinning shell. Of major importance is the fact that in the spinning case the equilibrium configuration, about which vibrations occur, is not a zero stress state. This pre-stressed condition gives rise to membrane restoring forces in addition to those associated with the usual shell theory. An additional complication arises because of the Coriolis acceleration coupling between the components of displacement which arises due to spin. Because of these additional difficulties most of the solution techniques available in the literature cannot be applied to the present situation.

Numerical approaches to the free vibration eigenvalue problem for spinning shells are required. Two distinct methods appear in the literature. In one method, most recently reported by Zarghamee and Robinson [26], values for the natural frequency are assumed and the equations of motion are integrated by a suitable technique to see if the boundary conditions can be satisfied. If they are not satisfied, a new value is assumed for the natural frequency and the process is repeated. The vanishing of a certain determinant which is a continuous function of the assumed frequency indicates the satisfaction of the boundary conditions. An iterative process based on the value of this determinant can be used to find the frequency at which it vanishes. In the second method, due to Cohen [27], the mode shape is iterated on and convergence is obtained when the sequence of frequency estimates based on successive mode estimates reaches a minimum. The former procedure is an extension of Holzer's method and the latter an extension of Stodola's method.

This research program deals with a broad range of problems within the general framework of the study of the static and dynamic characteristics of shallow spinning shells. The general equations of static and dynamic equilibrium are formulated from a variational principle. The equations of motion for free vibrations are obtained in terms of small perturbations about the equilibrium configuration assumed by the spinning shell. It is shown that the linear and nonlinear shallow shell theories of Reissner correspond to the present equations in the static case.

The spinning disk equations presented by Timoshenko and Goodier [28] are the limiting case of these shallow shell equations. The spinning membrane shell vibration equations of Johnson [8] can be shown to be identical with the vibration equations obtained in the present analysis. The equations of motion for the transverse vibrations of spinning membrane disks studied by Simmonds [7] and Eversman [12] and the equations of motion for the in-plane vibrations of spinning disks treated by Simmonds [29] are special cases of the results obtained in this investigation.

Several digital computer routines have been written to evaluate various aspects of the theory developed herein. Three programs have been written to calculate the static stress distribution in a spinning shallow shell. One program is for the calculation of the stress distribution in a freely spinning shell (no hub) using an analytically developed solution to Reissner's linear equations. The other two programs deal with the case of the shell with a fully clamped hub. One utilizes Reissner's linear theory and the other utilizes his nonlinear theory. These static stress calculations were carried out and are reported here since they must be available before a vibration analysis can be carried out. Of particular theoretical interest is the comparison of the linear and nonlinear theory and the shell geometries for which a nonlinear theory must be used and those for which a linear theory is adequate. In addition, the shell configurations for which a bending theory is required and a membrane theory is inadequate are clearly shown.

The results of Simmonds [7] and Eversman [12] for the natural frequencies of free transverse vibrations of spinning membrane disks have been extended in this investigation by the computation of results for the case of an annular membrane with a frictionless hub and for the most physically significant case, that of a fully clamped, or "built-in" hub configuration. The intermediate case of partial clamping, studied by Bulkeley and Savage [10] in the axisymmetric case, is also discussed in the general case.

The equations of motion for the shell vibrations have been formulated in a way appropriate for the determination of the natural frequencies by the method of Zarghamee and Robinson [26]. The procedure is much more complicated than their analysis since the equilibrium stress and displacement distribution must be computed prior to or during the iteration scheme and appears as a variable coefficient in the differential equations. No numerical results have been obtained due to the limitations of available computational equipment.

CHAPTER 2

EQUATIONS OF MOTION

2.1 Preliminary Considerations

We will be interested in deriving the equations of motion for the free vibrations of a shallow spherical shell which is spinning about its axis of symmetry. It will be assumed that the shell is isotropic and that the basic assumptions inherent in Reissner's linear and nonlinear theories [14,30] are valid. The vibrations will be considered to be small perturbations about the equilibrium configuration assumed by the shell when spinning. Since the equilibrium configuration must be established before the dynamic problem can be solved, the static and dynamic problems will be formulated separately. The equilibrium problem will be formulated allowing for finite rotations.

2.2 Axis System and Nomenclature

The axis system utilized in this analysis is identical to the one used by Reissner [30]. As seen in Figure 1, the equation describing the shallow spherical shell, measured from the base plane, is

$$z = \sqrt{R^2 - r^2} - (R - h) \tag{2.2.1}$$

and the local slope is

$$\frac{dz}{dr} = -\frac{r}{\sqrt{R^2 - r^2}} \approx -\frac{r}{R} \tag{2.2.2}$$

The approximation in Eq. (2.2.2) is the shallow shell approximation based on the assumption that

$$\frac{r}{R} << 1$$

The basic shell equations will be formulated in terms of displacements which are meridional, tangential, and normal, as shown in Figure 2. This is the convention used by Reissner in his linear theory [30]. In his nonlinear theory he uses displacements which are radial, tangential, and axial, as shown in Figure 3. The connection between the two systems, within the scope of shallow shell theory, is

$$\begin{aligned} w &= \bar{w} \\ u &= \bar{u} - \bar{w}\, r/R \\ v &= \bar{v} \end{aligned} \tag{2.2.3}$$

where the barred quantities refer to the system in which the deflections are radial, tangential, and axial.

2.3 Static Equilibrium Equations

The equilibrium configuration for the spinning shell is one of axial symmetry since the shell and load are both axially symmetric. The following fundamental relations, specialized for the axisymmetric case and consistent with the theory of shallow isotropic shells [14,30,42], will be used:

Strain-Displacement

$$\epsilon_r = u_r + \frac{w}{R} + \frac{1}{2} w_r^2 + \zeta K_r$$

$$\epsilon_\theta = \frac{u}{r} + \frac{w}{R} + \zeta K_\theta \qquad (2.3.1)$$

Stress-Strain

$$\sigma_r = \frac{E}{1-\nu^2}\left[\epsilon_r + \nu\epsilon_\theta\right]$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left[\epsilon_\theta + \nu\epsilon_r\right] \qquad (2.3.2)$$

Curvature Change-Displacement

$$K_r = -\omega_{,rr}$$

$$K_\theta = -\frac{1}{r}\omega_{,r} \qquad (2.3.3)$$

The co-ordinate ζ which appears in the strain-displacement relations is the distance from the mid-surface of the shell to a shell element, measured outward along the local normal.

The strain energy of the deformed shell is given by

$$U = \frac{1}{2}\iiint\left[\sigma_r\epsilon_r + \sigma_\theta\epsilon_\theta\right] r\,dr\,d\theta\,d\zeta \qquad (2.3.4)$$

If we write the strain-displacement expressions as

$$\epsilon_r = e_r + \zeta K_r$$

$$\epsilon_\theta = e_\theta + \zeta K_\theta \qquad (2.3.5)$$

where e_r and e_θ are mid-surface strains, we obtain for the strain energy

$$U = \frac{1}{2}\iiint \frac{E}{1-\nu^2}\left[e_r^2 + e_\theta^2 + 2\nu e_r e_\theta\right] r\,dr\,d\theta\,d\zeta$$

$$+\frac{1}{2}\iiint \zeta^2 \frac{E}{1-\nu^2}\left[K_r^2 + K_\theta^2 + 2\nu K_r K_\theta\right] r\,dr\,d\theta\,d\zeta$$

$$+\frac{1}{2}\iiint 2\zeta \frac{E}{1-\nu^2}\left[e_r K_r + e_\theta K_\theta + 2\nu(e_r K_\theta + e_\theta K_r)\right] r\,dr\,d\theta\,d\zeta \qquad (2.3.6)$$

If we integrate on ζ and note that e_r, e_θ, κ_r, κ_θ, being mid-surface strains and curvature changes, are not functions of ζ, we obtain

$$U = \frac{tE}{2(1-\nu^2)}\iint\left[e_r^2 + e_\theta^2 + 2\nu e_r e_\theta\right] r\,dr\,d\theta$$

$$+ \frac{D}{2}\iint\left[K_r^2 + K_\theta^2 + 2\nu K_r K_\theta\right] r\,dr\,d\theta \qquad (2.3.7)$$

where

$$D = \frac{Et^3}{12(1-\nu^2)}$$

and t is the shell thickness, assumed constant.

As shown in Figure 4, the inertia load of an element of the shell is horizontal and given by

$$F_I = m\omega^2\left(r + u + w\frac{r}{R}\right) t\, r\, dr\, d\theta$$

where m is the constant mass density of the shell and ω is the spin angular velocity. The corresponding potential for the element is

$$dV_I = -\frac{1}{2} m\omega^2\left(r + u + w\frac{r}{R}\right)^2 t\, r\, dr\, d\theta$$

By combining the strain energy of the shell and the potential energy of the load, the total potential energy of the shell is given by

$$V = \frac{t}{2}\iint\Big\{\frac{E}{1-\nu^2}\left[e_r^2 + e_\theta^2 + 2\nu e_r e_\theta\right] + \frac{D}{t}\left[K_r^2 + K_\theta^2 + 2\nu K_r K_\theta\right] - m\omega^2\left(r + u + w\frac{r}{R}\right)^2\Big\} r\, dr\, d\theta \qquad (2.3.8)$$

By introducing the strain-displacement relations and using the principle of minimum potential energy we obtain the variational problem

$$\delta \iint \Big[\frac{E}{1-\nu^2} \Big\{ \Big[u_r + \frac{w}{R} + \frac{1}{2} w_r^2 \Big]^2 + \Big[\frac{u}{r} + \frac{w}{R} \Big]^2$$

$$+ 2\nu \Big[u_r + \frac{w}{R} + \frac{1}{2} w_r^2 \Big] \Big[\frac{u}{r} + \frac{w}{R} \Big] \Big\} + \frac{D}{t} \Big\{ w_{rr}^2$$

$$+ \Big(\frac{1}{r} w_r \Big)^2 + 2\nu \frac{w_{rr} w_r}{r} \Big\} - m\omega^2 \Big(r + u + w\frac{r}{R} \Big)^2 \Big] r\,dr\,d\theta = 0 \qquad (2.3.9)$$

The Euler equations for this variational problem are [32]

$$\frac{d}{dr}(r\sigma_r) - \sigma_\theta + m\omega^2 \Big(r + u + w\frac{r}{R} \Big) r = 0 \qquad (2.3.10)$$

$$\frac{D}{t} \nabla^2 \nabla^2 w - \frac{1}{r}\frac{d}{dr}(r\sigma_r w_r)$$

$$+ \frac{1}{R}(\sigma_r + \sigma_\theta) - m\omega^2 \Big(r + u + w\frac{r}{R} \Big) \frac{r}{R} = 0 \qquad (2.3.11)$$

where σ_r and σ_θ are the mid-surface stresses. The boundary conditions to be satisfied will consist of the hub conditions on some hub radius and the natural boundary conditions on the free outer edge, which are

$$w_{rr} + \nu \frac{w_r}{r} = 0$$

$$\frac{d}{dr}(\nabla^2 w) = 0$$

$$\sigma_r = 0 \tag{2.3.12}$$

These conditions correspond respectively to the vanishing of the bending moment, the Kirchoff condition, and the vanishing of the meridional stress at the free edge.

2.4 Reduction of the General Equations to Special Cases

A. The Spinning Flat Disk

In this case Eq. (2.3.10) is the only one applicable since when R becomes infinite there is no transverse load. Hence as $R \to \infty$ we obtain

$$\frac{d}{dr}(r\sigma_r) - \sigma_\theta + m\omega^2 r(r+u) = 0 \tag{2.4.1}$$

which corresponds to the result obtained by Simmonds [29]. In the case when we assume

$$\frac{u}{r} \ll 1$$

we obtain

$$\frac{d}{dr}(r\sigma_r) - \sigma_\theta + m\omega^2 r^2 = 0 \tag{2.4.2}$$

which corresponds to the governing equation derived by Timoshenko and Goodier [28]. Simmond's equations for the finite deflection of a normally loaded spinning membrane can be obtained from Eqs. (2.4.2) and (2.3.10) by allowing $R \to 0, D \to 0$ and replacing the inertia load in Eq. (2.3.11) by a normal pressure p to obtain

$$\frac{d}{dr}(r\sigma_r) - \sigma_\theta + m\omega^2 r^2 = 0$$

$$\frac{1}{r}\frac{d}{dr}(r\sigma_r w_r) = p \tag{2.4.3}$$

B. The Linear Reissner Equations

The linear Reissner equations in the axisymmetric case can be obtained by neglecting the non-linear term in Eq. (2.3.11) which arises because of the inclusion of the effects of finite rotations, and by assuming

$$\frac{1}{r}\left(u + w\frac{r}{R}\right) \ll 1$$

With these conditions we obtain

$$\frac{d}{dr}(r\sigma_r) - \sigma_\theta + m\omega^2 r^2 = 0 \qquad (2.4.4)$$

$$\frac{D}{t}\nabla^2\nabla^2 w + \frac{1}{R}(\sigma_r + \sigma_\theta) - m\omega^2\frac{r^2}{R} = 0 \qquad (2.4.5)$$

Equation (2.4.5) corresponds to Reissner's Eq. (16) in the axisymmetric case [30], with

$$p = \frac{m\omega^2 r^2}{R}$$

Equation (2.4.4) corresponds to Reissner's Equation (3a) in the axisymmetric case with

$$p_r = m\omega^2 r$$

C. Reissner's Nonlinear Equations

Reissner's shallow shell equations which include the effects of finite rotations [14] can be obtained from the present theory by introducing a stress function and deformation variable, and properly accounting for the transition from displacements in the meridional, tangential, and normal directions, to displacements in the radial, tangential, and axial directions.

Prior to obtaining the equations of equilibrium, the appropriate strain-displacement relations for the pertinent type of displacements can be obtained from Eqs. (2.3.1) and the transformations of Eqs. (2.2.3):

$$\epsilon_r = u_r - \frac{r}{R}\,w_r + \frac{1}{2}\,w_r^2 + \zeta K_r$$

$$\epsilon_\theta = \frac{u}{r} + \zeta K_\theta \tag{2.4.6}$$

Here we have not used the barred quantities to distinguish the co-ordinate system.

Equation (2.3.10), with the assumption

$$\frac{1}{r}\left(u + w\frac{r}{R}\right) \ll 1 \tag{2.4.7}$$

is equivalent to Reissner's Eq. (2.21). In the present notation we have

$$\frac{d}{dr}(r\sigma_r) - \sigma_\theta + m\,\omega^2 r^2 = 0$$

This implies the existence of a stress function such that

$$r\,\sigma_r = \psi$$

$$\sigma_\theta = \frac{d\psi}{dr} + m\omega^2 r^2$$

The differential equation which ψ satisfies can be obtained from the appropriate compatibility equation. It is not difficult to verify that we must have for the mid-surface strains [14]

$$r\frac{de_\theta}{dr} + e_\theta - e_r + \frac{1}{2}\,w_r^2 - \frac{r}{R}\,w_r = 0$$

By making use of the stress-strain relations

$$e_r = \frac{1}{E}\left[\sigma_r - \nu\sigma_\theta\right]$$

$$e_\theta = \frac{1}{E}\left[\sigma_\theta - \nu\sigma_r\right]$$

and the definition of the stress function we obtain

$$\frac{1}{E}\left[r\psi_{rr} + \psi_r - \frac{\psi_r}{r} + (3+\nu)m\omega^2 r^2\right] + \frac{1}{2}\,w_r^2 - \frac{r}{R}\,w_r = 0$$

If we introduce a displacement variable ϕ such that

$$\frac{dw}{dr} = \phi$$

there results

$$\frac{1}{E}\left[\Psi_{rr}+\frac{\Psi_r}{r}-\frac{\Psi_r}{r^2}\right]+\frac{1}{2}\frac{\Phi^2}{r}-\frac{\Phi}{r}=-\frac{(3+\nu)m\omega^2 r}{E} \tag{2.4.8}$$

A second equation is obtained from Eq. (2.3.11) by introducing the stress function and displacement variable and employing the approximation of Eq. (2.4.7):

$$\frac{1}{r}\frac{d}{dr}\left\{\frac{D}{t}\left[r\Phi_{rr}+\Phi_r-\frac{\Phi}{r}\right]+\frac{r\Psi}{R}-\Phi\Psi\right\}=0$$

Upon integration we obtain the second governing equation

$$\frac{D}{t}\left\{\Phi_{rr}+\frac{\Phi_r}{r}-\frac{\Phi}{r^2}\right\}+\frac{\Psi}{R}-\frac{\Phi\Psi}{r}=0 \tag{2.4.9}$$

where the constant of integration has been shown to be zero because of the free edge conditions.

It is of interest to note that Eqs. (2.4.8) and (2.4.9) require only four boundary conditions. The vertical edge reaction at the outer edge is automatically satisfied and no condition on ω is required unless the displacement is desired from a further integration of Φ.

A second form of Eqs. (2.4.8) and (2.4.9) in terms of stress resultants rather than stresses is

$$\frac{1}{Et}\left[\psi_{rr} + \frac{\psi_r}{r} - \frac{\psi}{r^2}\right] + \frac{1}{2}\frac{\phi^2}{r} - \frac{\phi}{r} = -\frac{(3+\nu)\rho\omega^2 r}{Et}$$

$$D\left[\phi_{rr} + \frac{\phi_r}{r} - \frac{\phi}{r^2}\right] + \frac{\psi}{R} - \frac{\phi\psi}{r} = 0$$

where

$$r t\sigma_r = rN_{rr} = \psi$$

$$t\sigma_\theta = N_{\theta\theta} = \frac{d\psi}{dr} + \rho\omega^2 r^2$$

and

$$\rho = \text{MASS PER UNIT AREA} = mt$$

2.5 Equations of Motion for Small Free Vibrations About the Equilibrium Configuration

The equations of motion for the free vibrations of the spinning shallow spherical shell will be obtained by considering small perturbations about the equilibrium configuration assumed by the spinning shell. The dynamic theory will be based on the same assumptions as the static theory with the exceptions that the perturbations will have infinitesimal rotations and the possibility of asymmetric motions will be included.

The fundamental relations to be used are the following:

Strain-Displacement

$$\epsilon_r = u_r + \frac{w}{R} + \frac{1}{2} w_r^2 + \zeta \kappa_r = e_r + \zeta \kappa_r$$

$$\epsilon_\theta = \frac{u}{r} + \frac{v_\theta}{r} + \frac{w}{R} + \frac{1}{2}\left(\frac{w_\theta}{r}\right)^2 + \zeta \kappa_\theta = e_\theta + \zeta \kappa_\theta$$

$$\gamma_{r\theta} = \frac{u_\theta}{r} + v_r - \frac{v}{r} + \frac{w_r w_\theta}{r} + \zeta \kappa_{r\theta} = g_{r\theta} + \zeta \kappa_{r\theta} \tag{2.5.1}$$

Stress-Strain

$$\sigma_r = \frac{E}{1-\nu^2}\left[\epsilon_r + \nu \epsilon_\theta\right]$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left[\epsilon_\theta + \nu \epsilon_r\right]$$

$$\tau_{r\theta} = G \gamma_{r\theta} = \frac{E}{2(1+\nu)} \gamma_{r\theta} \tag{2.5.2}$$

Curvature Change-Displacement

$$\kappa_r = - w_{rr}$$

$$\kappa_\theta = -\frac{1}{r} w_r - \frac{1}{r^2} w_{\theta\theta}$$

$$\kappa_{r\theta} = -2 \frac{\partial}{\partial r}\left(\frac{1}{r} w_\theta\right) \tag{2.5.3}$$

The strain energy of the deformed shell is

$$U = \frac{1}{2}\iiint\left[\sigma_r\epsilon_r + \sigma_\theta\epsilon_\theta + \tau_{r\theta}\gamma_{r\theta}\right] r\,dr\,d\theta\,d\zeta$$

In terms of strains alone this becomes

$$U = \frac{1}{2}\iiint\left\{\frac{E}{1-\nu^2}\left[\epsilon_r^2 + \epsilon_\theta^2 + 2\nu\epsilon_r\epsilon_\theta\right] + G\gamma_{r\theta}^2\right\} r\,dr\,d\theta\,d\zeta$$

In the static case the inertia load entered the formulation as an equivalent applied load and hence as a contributor to the total shell potential energy. In the dynamic case the inertia load appears naturally in the kinetic energy and should not be considered separately. By following the development in the static equilibrium case we obtain for the total potential energy

$$V = \iint\left[\frac{tE}{2(1-\nu^2)}\left(e_r^2 + e_\theta^2 + 2\nu e_r e_\theta + G g_{r\theta}^2\right)\right.$$

$$+ \frac{D}{2}\left\{\omega_{rr}^2 + \left(\frac{1}{r}\omega_r + \frac{1}{r^2}\omega_{\theta\theta}\right)^2 + 2\nu\omega_{rr}\left(\frac{1}{r}\omega_r + \frac{1}{r^2}\omega_{\theta\theta}\right)\right.$$

$$\left.\left. + \frac{1-\nu}{2}\left[2\frac{\partial}{\partial r}\left(\frac{1}{r}\omega_\theta\right)\right]^2\right\}\right] r\,dr\,d\theta$$

(2.5.4)

where e_r, e_θ, $g_{r\theta}$ are the strains in the shell mid-surface.

To obtain the kinetic energy we must derive the expressions for the absolute velocity components of a shell element. In an axis system fixed to the shell and oriented with axes in the local meridional, tangential, and normal directions the velocity of a shell element relative to the shell is

$$\vec{V}_R = \dot{u}\,\vec{e}_\varphi + \dot{v}\,\vec{e}_\theta + \dot{w}\,\vec{e}_r$$

where $\bar{e}_\varphi$, $\bar{e}_\theta$, $\bar{e}_r$ are unit vectors in the meridional, tangential, and radial directions, respectively. The angular velocity, in the same axis system is

$$\vec{\omega} = -\omega\frac{r}{R}\,\vec{e}_\varphi + \omega\,\vec{e}_r$$

where the shallow shell approximation has been used in resolving the components. The position vector of a displaced mass element is

$$\vec{\delta} = u\,\vec{e}_\varphi + v\,\vec{e}_\theta + w\,\vec{e}_r$$

and the velocity of the origin of the axis system (coincides with the undeformed element) is

$$\vec{V}_o = r\,\omega\,\vec{e}_\theta$$

The absolute velocity of an element is

$$\vec{V} = \vec{V}_o + \vec{V}_R + \vec{\omega} \times \vec{\delta}$$

which yields

$$\vec{V} = (\dot{u} - \omega v)\,\vec{e}_\phi + \left[\dot{v} + \left(r + u + w\frac{r}{R}\right)\omega\right]\vec{e}_\theta + \left(\dot{w} - v\frac{r}{R}\omega\right)\vec{e}_r \tag{2.5.5}$$

The kinetic energy of the deforming shell is then written

$$T = \frac{t}{2}\iint m\left\{\left[\dot{u} - \omega v\right]^2 + \left[\dot{v} + \left(r + u + w\frac{r}{R}\right)\omega\right]^2 + \left[\dot{w} - v\frac{r}{R}\omega\right]^2\right\} r\,dr\,d\theta \tag{2.5.6}$$

The governing equations of motion are obtained from Hamilton's principle:

$$\delta\int_{t_1}^{t_2}(T - V)\,dt = 0$$

By replacing e_r, e_θ, and $g_{r\theta}$ in Eq. (2.5.4) by th
definitions from Eqs. (2.5.1), we obtain for the
problem

$$\delta \iiint \Big[m\Big\{ [\dot{u} - v\omega]^2 + [\dot{v} + (r + u + w\tfrac{r}{R})\omega]^2 + [\dot{w} - $$

$$- \frac{E}{1-\nu^2}\Big\{ \Big[u_r + \frac{w}{R} + \frac{1}{2} w_r^2\Big] + \Big[\frac{v_\theta}{r} + \frac{u}{r} + \frac{w}{R} + \frac{1}{2}\Big(\frac{w}{r}$$

$$+ 2\nu \Big[u_r + \frac{w}{R} + \frac{1}{2} w_r^2\Big]\Big[\frac{v_\theta}{r} + \frac{u}{r} + \frac{w}{R} + \frac{1}{2}\Big(\frac{w_\theta}{r}\Big)^2\Big]$$

$$- G\Big\{\Big[v_r + \frac{u_\theta}{r} - \frac{v}{r} + \frac{w_r w_\theta}{r}\Big]^2\Big\} - \frac{D}{t}\Big\{ w_{rr}^2$$

$$+ \Big(\frac{1}{r} w_r + \frac{1}{r^2} w_{\theta\theta}\Big)^2 + 2\nu\, w_{rr}\Big(\frac{1}{r} w_r + \frac{1}{r^2} w_{\theta\theta}\Big)$$

$$+ \frac{1-\nu}{2}\Big[2 \frac{\partial}{\partial r}\Big(\frac{1}{r} w_\theta\Big)\Big]^2 \Big\}\Big]\, r\, dr\, d\theta\, dt = 0$$

The shell displacements will be taken in the form

$$
\begin{aligned}
u(r,\theta,t) &= u_o(r) + u^*(r,\theta,t) \\
v(r,\theta,t) &= v^*(r,\theta,t) \\
w(r,\theta,t) &= w_o(r) + w^*(r,\theta,t)
\end{aligned}
$$

Where $u_o(r)$, $w_o(r)$ are the shell displacements in the equilibrium configuration of spin about the shell axis and u^*, v^*, w^* are assumed to be small perturbations about the equilibrium position. If these expressions for u, v, and w are inserted into Eq. (2.5.7) and the result is rearranged slightly to group terms involving the equilibrium displacements, we obtain

$$
\delta\left\{\iiint\left[m\omega^2\left[r+u_o+w_o\frac{r}{R}\right]^2 - \frac{E}{1-\nu^2}\left\{\left[u_{or}+\frac{w_o}{R}+\frac{1}{2}w_{or}^2\right]^2\right.\right.\right.
$$

$$
\left.+\left[\frac{u_o}{r}+\frac{w_o}{R}\right]^2+2\nu\left[u_{or}+\frac{w_o}{R}+\frac{1}{2}w_{or}^2\right]\left[\frac{u_o}{r}+\frac{w_o}{R}\right]\right\}
$$

$$
\left.\left.-\frac{D}{t}\left\{w_{orr}^2+\left(\frac{1}{r}w_{or}\right)^2+\frac{2\nu}{r}w_{orr}w_{or}\right\}\right]r\,dr\,d\theta\,dt\right\}
$$

$$
+2\delta\left\{\iiint\left[m\left\{\left[r+u_o+w_o\frac{r}{R}\right]\omega\left[\dot{v}^*+\left(u^*+w^*\frac{r}{R}\right)\omega\right]\right\}\right.\right.
$$

$$
-\frac{E}{1-\nu^2}\left\{\left[u_{or}+\frac{w_o}{R}+\frac{1}{2}w_{or}^2\right]\left[w_{or}w_r^*+u_r^*+\frac{w^*}{R}+\frac{1}{2}w_r^{*2}\right]\right.
$$

By replacing e_r, e_θ, and $g_{r\theta}$ in Eq. (2.5.4) by their definitions from Eqs. (2.5.1), we obtain for the variational problem

$$\delta \iiint \Big[m\Big\{ [\dot{u} - v\omega]^2 + [\dot{v} + (r + u + w\tfrac{r}{R})\omega]^2 + [\dot{w} - v\tfrac{r}{R}\omega]^2 \Big\}$$

$$- \frac{E}{1-\nu^2}\Big\{ [u_r + \tfrac{w}{R} + \tfrac{1}{2} w_r^2] + [\tfrac{v_\theta}{r} + \tfrac{u}{r} + \tfrac{w}{R} + \tfrac{1}{2}(\tfrac{w_\theta}{r})^2]^2$$

$$+ 2\nu [u_r + \tfrac{w}{R} + \tfrac{1}{2} w_r^2][\tfrac{v_\theta}{r} + \tfrac{u}{r} + \tfrac{w}{R} + \tfrac{1}{2}(\tfrac{w_\theta}{r})^2] \Big\}$$

$$- G\Big\{ [v_r + \tfrac{u_\theta}{r} - \tfrac{v}{r} + \tfrac{w_r w_\theta}{r}]^2 \Big\} - \frac{D}{t}\Big\{ w_{rr}^2$$

$$+ (\tfrac{1}{r} w_r + \tfrac{1}{r^2} w_{\theta\theta})^2 + 2\nu w_{rr}(\tfrac{1}{r} w_r + \tfrac{1}{r^2} w_{\theta\theta})$$

$$+ \frac{1-\nu}{2}[2 \tfrac{\partial}{\partial r}(\tfrac{1}{r} w_\theta)]^2 \Big\}\Big] r\,dr\,d\theta\,dt = 0$$

(2.5.7)

The shell displacements will be taken in the form

$$
\begin{aligned}
u(r,\theta,t) &= u_0(r) + u^*(r,\theta,t) \\
v(r,\theta,t) &= v^*(r,\theta,t) \\
w(r,\theta,t) &= w_0(r) + w^*(r,\theta,t)
\end{aligned}
$$

Where $u_0(r)$, $w_0(r)$ are the shell displacements in the equilibrium configuration of spin about the shell axis and u^*, v^*, w^* are assumed to be small perturbations about the equilibrium position. If these expressions for u, v, and w are inserted into Eq. (2.5.7) and the result is rearranged slightly to group terms involving the equilibrium displacements, we obtain

$$
\delta\Big\{\iiint\Big[m\omega^2\Big[r+u_0+w_0\frac{r}{R}\Big]^2 - \frac{E}{1-\nu^2}\Big\{\Big[u_{0r}+\frac{w_0}{R}+\frac{1}{2}w_{0r}^2\Big]^2
$$

$$
+\Big[\frac{u_0}{r}+\frac{w_0}{R}\Big]^2 + 2\nu\Big[u_{0r}+\frac{w_0}{R}+\frac{1}{2}w_{0r}^2\Big]\Big[\frac{u_0}{r}+\frac{w_0}{R}\Big]\Big\}
$$

$$
-\frac{D}{t}\Big\{w_{0rr}^2+\Big(\frac{1}{r}w_{0r}\Big)^2+\frac{2\nu}{r}w_{0rr}w_{0r}\Big\}\Big]r\,dr\,d\theta\,dt\Big\}
$$

$$
+2\delta\Big\{\iiint\Big[m\Big\{\Big[r+u_0+w_0\frac{r}{R}\Big]\omega\Big[\dot{v}^*+\Big(u^*+w^*\frac{r}{R}\Big)\omega\Big]\Big\}
$$

$$
-\frac{E}{1-\nu^2}\Big\{\Big[u_{0r}+\frac{w_0}{R}+\frac{1}{2}w_{0r}^2\Big]\Big[w_{0r}w_r^*+u_r^*+\frac{w^*}{R}+\frac{1}{2}w_r^{*2}\Big]
$$

$$+\left[\frac{u_o}{r}+\frac{w_o}{R}\right]\left[\frac{v_\theta^*}{r}+\frac{u^*}{r}+\frac{w^*}{R}+\frac{1}{2}\left(\frac{w_\theta^*}{r}\right)^2\right]$$

$$+\nu\left[\frac{u_o}{r}+\frac{w_o}{R}\right]\left[w_{o_r}w_r^*+u_r^*+\frac{w^*}{R}+\frac{1}{2}w_r^{*2}\right]$$

$$+\nu\left[u_{o_r}+\frac{w_o}{R}+\frac{1}{2}w_{o_r}^2\right]\left[\frac{v_\theta^*}{r}+\frac{u^*}{r}+\frac{w^*}{R}+\frac{1}{2}\left(\frac{w_\theta^*}{r}\right)^2\right]\Big\}$$

$$-\frac{D}{t}\Big\{w_{o_{rr}}w_{rr}^*+\frac{1}{r}w_{o_r}\left(\frac{1}{r}w_r^*+\frac{1}{r^2}w_{\theta\theta}^*\right)$$

$$+\nu w_{o_{rr}}\left(\frac{1}{r}w_r^*+\frac{1}{r^2}w_{\theta\theta}^*\right)\quad+\frac{\nu}{r}w_{o_r}w_{rr}^*\Big\}\Big]\,r\,dr\,d\theta\,dt\Big\}$$

$$+\delta\Big\{\iiint\Big[m\Big\{\left[\dot{w}^*-v^*\frac{r}{R}\omega\right]^2+\left[\dot{u}^*-v^*\omega\right]^2+\left[\dot{v}^*+\left(u^*+w^*\frac{r}{R}\right)\omega\right]^2\Big\}$$

$$-\frac{E}{1-\nu^2}\Big\{\left[u_r^*+\frac{w^*}{R}+w_{o_r}w_r^*+\frac{1}{2}w_r^{*2}\right]^2+\left[\frac{v_\theta^*}{r}+\frac{u^*}{r}+\frac{w^*}{R}+\frac{1}{2}\left(\frac{w_\theta^*}{r}\right)^2\right]^2$$

$$+2\nu\left[u_r^*+\frac{w^*}{R}+w_{o_r}w_r^*+\frac{1}{2}w_r^{*2}\right]\left[\frac{v_\theta^*}{r}+\frac{u^*}{r}+\frac{w^*}{R}+\frac{1}{2}\left(\frac{w_\theta^*}{r}\right)^2\right]\Big\}$$

$$-G\Big\{\left[v_r^*+\frac{u_\theta^*}{r}-\frac{v^*}{r}+\frac{w_\theta^*(w_{o_r}+w_r^*)}{r}\right]^2\Big\}-\frac{D}{t}\Big\{w_{rr}^{*2}$$

$$+\left(\frac{1}{r}w_r^*+\frac{1}{r^2}w_{\theta\theta}^*\right)^2+2\nu w_{rr}^*\left(\frac{1}{r}w_r^*+\frac{1}{r^2}w_{\theta\theta}^*\right)$$

$$+\frac{1-\nu}{2}\left[2\frac{\partial}{\partial r}\left(\frac{1}{r}w_\theta^*\right)\right]^2\Big\}\Big]\,r\,dr\,d\theta\,dt\Big\}=0 \qquad (2.5.8)$$

By referring to Eq. (2.3.9), which defines the equilibrium configuration, and by using the fact that the processes of variation and integration can be interchanged, it is noted that the variation of the first integral in Eq. (2.5.8) vanishes. To interpret the second integral term in the variation we note that the strain-displacement relations of Eqs. (2.5.1) become

$$\epsilon_r = u_{or} + \frac{w_o}{R} + \frac{1}{2} w_{or}^2 - \zeta w_{orr} + u_r^* + \frac{w^*}{R} + w_{or} w_r^* + \frac{1}{2} w_r^{*2} - \zeta w_{rr}^*$$

$$\epsilon_\theta = \frac{u_o}{r} + \frac{w_o}{R} - \zeta \frac{w_{or}}{r} + \frac{v_\theta^*}{r} + \frac{u^*}{r} + \frac{w^*}{R} + \frac{1}{2}\left(\frac{w_\theta^*}{r}\right)^2 - \frac{\zeta}{r} w_r^* - \frac{\zeta}{r^2} w_{\theta\theta}^*$$

$$\gamma_{r\theta} = \frac{u_\theta^*}{r} + v_r^* - \frac{v^*}{r} + \frac{(w_{or} + w_r^*)\, w_\theta^*}{r} - 2\zeta \frac{\partial}{\partial r}\left(\frac{1}{r} w_\theta^*\right)$$

We note that the strains can be broken down into the sum of equilibrium strains and perturbation strains

$$\epsilon_r = \epsilon_{r_o} + \epsilon_r^*$$

$$\epsilon_\theta = \epsilon_{\theta_o} + \epsilon_\theta^*$$

$$\gamma_{r\theta} = \gamma_{r\theta}^*$$

where, as previously defined, the equilibrium strains are

$$\epsilon_{r_o} = u_{or} + \frac{w_o}{R} + \frac{1}{2} w_{or}^2 - \zeta w_{orr}$$

$$\epsilon_{\theta_o} = \frac{u_o}{r} + \frac{w_o}{R} - \zeta \frac{w_{or}}{r}$$

The perturbation strains, defined as the additional strains due to perturbation of the shell from its equilibrium configuration, are

$$\epsilon_r^* = u_r^* + \frac{w^*}{R} + w_{0_r} w_r^* + \frac{1}{2} w_r^{*2} - \varsigma w_{rr}^*$$

$$\epsilon_\theta^* = \frac{v_\theta^*}{r} + \frac{u^*}{r} + \frac{w^*}{R} + \frac{1}{2}\left(\frac{w_\theta^*}{r}\right)^2 - \frac{\varsigma}{r} w_r^* - \frac{\varsigma}{r^2} w_{\theta\theta}^*$$

$$\gamma_{r\theta}^* = \frac{u_\theta^*}{r} + v_r^* - \frac{v^*}{r} + \frac{(w_{0_r} + w_r^*)}{r} w_\theta^* - 2\varsigma \frac{\partial}{\partial r}\left(\frac{1}{r} w_\theta^*\right)$$

The second integral term can then be written

$$2\delta \iiint \Big[m\Big\{\omega\Big[r + u_0 + w_0 \frac{r}{R}\Big]\Big[\dot{v}^* + \Big(u^* + w^* \frac{r}{R}\Big)\omega\Big]\Big\}$$

$$- \Big\{\sigma_{r_0}\epsilon_r^* + \sigma_{\theta_0}\epsilon_r^*\Big\} - \frac{D}{t}\Big\{w_{0_{rr}}\Big[w_{rr}^* + \nu\Big(\frac{1}{r} w_r^* + \frac{1}{r^2} w_{\theta\theta}^*\Big)\Big]$$

$$+ \frac{1}{r} w_{0_r}\Big[\frac{1}{r} w_r^* + \frac{1}{r^2} w_{\theta\theta}^* + \nu w_{rr}^*\Big]\Big\} r\, dr\, d\theta\, dt$$

This is recognized as a variation problem involving the equilibrium loads and stresses acting on the perturbation strains and displacements. As such, the variation must vanish. Hence, the variation problem reduces to

$$\delta\iiint\Big[m\Big\{\big[\dot{w}^* - v^*\tfrac{r}{R}\omega\big]^2+\big[\dot{u}^*-v^*\omega\big]^2+\big[\dot{v}^*+(u^*+w^*\tfrac{r}{R})\omega\big]^2\Big\}$$

$$-\frac{E}{1-\nu^2}\Big\{\Big[u_r^*+\frac{w^*}{R}+w_{0r}w_r^*+\frac{1}{2}w_r^{*2}\Big]^2+\Big[\frac{v_\theta^*}{r}+\frac{u^*}{r}+\frac{w^*}{R}+\frac{1}{2}\Big(\frac{w_\theta^*}{r}\Big)^2\Big]^2$$

$$+2\nu\Big[u_r^*+\frac{w^*}{R}+w_{0r}w_r^*+\frac{1}{2}w_r^{*2}\Big]\Big[\frac{v_\theta^*}{r}+\frac{u^*}{r}+\frac{w^*}{R}+\frac{1}{2}\Big(\frac{w_\theta^*}{r}\Big)^2\Big]\Big\}$$

$$-G\Big\{\Big[v_r^*+\frac{u_\theta^*}{r}-\frac{v^*}{r}+\frac{w_\theta^*(w_{0r}+w_r^*)}{r}\Big]^2\Big\}-\frac{D}{t}\Big\{w_{rr}^{*2}$$

$$+\Big(\frac{1}{r}w_r^*+\frac{1}{r^2}w_{\theta\theta}^*\Big)^2+2\nu\,w_{rr}^*\Big(\frac{1}{r}w_r^*+\frac{1}{r^2}w_{\theta\theta}^*\Big)$$

$$+\frac{1-\nu}{2}\Big[2\frac{\partial}{\partial r}\Big(\frac{1}{r}w_\theta^*\Big)\Big]^2\Big\}\Big]\,r\,dr\,d\theta\,dt=0$$

The Euler equations for this variational problem are [32]

$$\frac{\partial}{\partial r}(r\sigma_r^*)-\sigma_\theta^*+\frac{\partial\tau_{r\theta}^*}{\partial\theta}=mr\Big[\ddot{u}^*-2\omega\dot{v}^*-\Big(u^*+w^*\frac{r}{R}\Big)\omega^2\Big] \tag{2.5.9}$$

$$\frac{\partial}{\partial r}(r\tau_{r\theta}^*)+\frac{\partial\sigma_\theta^*}{\partial\theta}+\tau_{r\theta}^*=mr\Big[\ddot{v}^*+2\omega\Big(\dot{u}^*+\dot{w}^*\frac{r}{R}\Big)-v^*\omega^2\Big] \tag{2.5.10}$$

$$\frac{D}{t}\nabla^2\nabla^2 w^* - \frac{1}{r}\frac{\partial}{\partial r}\left\{ r\sigma_r^* w_{0r} + r\sigma_{r_0} w_r^* \right\}$$

$$-\frac{1}{r}\frac{\partial}{\partial \theta}\left\{ \frac{1}{r}\sigma_{\theta_0} w_\theta^* + w_{0r}\tau_{r\theta}^* \right\} + \frac{1}{R}(\sigma_r^* + \sigma_\theta^*)$$

$$= -m\left[\ddot{w}^* - 2\omega\dot{v}^*\frac{r}{R} - (u^* + w^*\frac{r}{R})\omega^2\frac{r}{R} \right] \tag{2.5.11}$$

where σ_r^*, σ_θ^*, $\tau_{r\theta}^*$ are mid-surface perturbation stresses and where, in Eq. (2.5.10), the shallow shell approximation $\frac{r}{R} \ll 1$, was used. The boundary conditions for this set of partial differential equations will consist of the hub conditions at the hub radius and the free edge conditions at the outer edge of the disk. The dynamic boundary conditions are the same as static boundary conditions and are

$r = a$:

$$u(a) = 0$$

$$v(a) = 0$$

$$w(a) = 0$$

$$w_r(a) = 0$$

$r = b$:

$$w_{rr} + \nu\left(\frac{1}{r}w_r + \frac{1}{r^2}w_{\theta\theta}\right) = 0$$

$$\frac{\partial}{\partial r}\nabla^2 w + \frac{1-\nu}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial^2 w}{\partial \theta^2}\right) = 0$$

$$\sigma_r(b) = 0$$

$$\tau_{r\theta}(b) = 0$$

where $r=a$ is the hub radius and $r=b$ is the shell outer radius. In terms of the perturbation quantities these become:

$r = a$:

$$u^*(a) = v^*(a) = w^*(a) = 0$$

$$w_r^*(a) = 0$$

$r = b$:

$$w_{rr}^* + \nu\left(\frac{1}{r} w_r^* + \frac{1}{r^2} w_{\theta\theta}^*\right)\Big/_{r=b} = 0$$

$$\frac{\partial}{\partial r}\nabla^2 w^* + \frac{1-\nu}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial^2 w^*}{\partial\theta^2}\right)\Big/_{r=b} = 0$$

$$\sigma_r^*(b) = \tau_{r\theta}^*(b) = 0$$

It should be noted that the displacements are referred to the unstressed configuration and hence the perturbation quantities are in directions established by the shape of the shell before it is spun. Simmonds [1] has circumvented this problem to some degree by establishing an intermediate reference frame associated with the shell equilibrium configuration and perturbations can be referred to this system. For vibration problems of shallow shells it is not felt that this refinement is justified.

2.6 Reduction of the Differential Equations for Special Cases

A. The Transverse Vibrations of a Flat Spinning

Circular Plate

In this case the radius of curvature is infinite and the equilibrium deflection in the normal, or transverse, direction vanishes. Equation (2.5.11) decouples from Eqs. (2.5.9) and (2.5.10) and becomes

$$\frac{D}{t}\nabla^2\nabla^2 w^* - \frac{1}{r}\frac{\partial}{\partial r}\left(r\sigma_{r_0} w_r^*\right) - \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\sigma_{\theta_0}}{r} w_\theta^*\right) = -m\ddot{w}^* \tag{2.6.1}$$

The boundary conditions are

$$w^*(a) = 0$$

$$w_r^*(a) = 0$$

$$w_{rr}^* + \nu\left(\frac{1}{r}w_r^* + \frac{1}{r^2}w_{\theta\theta}^*\right)\Big|_{r=b} = 0$$

$$\left[\frac{\partial}{\partial r}\left(\nabla^2 w\right) + \frac{1-\nu}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}w_{\theta\theta}^*\right)\right]\Big|_{r=b} = 0$$

B. In-Plane Vibrations of a Flat Spinning Circular Plate

As noted in (A), in the case of a flat plate the transverse and in-plane vibrations decouple. In this case, with the radius of curvature infinite, the in-plane vibrations are governed by Eqs. (2.5.9) and (2.5.10)

$$\frac{\partial}{\partial r}(r\sigma_r^*) - \sigma_\theta^* + \frac{\partial \tau_{r\theta}^*}{\partial \theta} = mr\left[\ddot{u}^* - 2\omega\dot{v}^* - u^*\omega^2\right] \tag{2.6.2}$$

$$\frac{\partial}{\partial r}(r\tau_{r\theta}^*) + \frac{\partial \sigma_\theta^*}{\partial \theta} + \tau_{r\theta}^* = mr\left[\ddot{v}^* + 2\omega\dot{u}^* - v^*\omega^2\right] \tag{2.6.3}$$

with boundary conditions in the clamped hub case

$$u^*(a) = 0$$
$$v^*(a) = 0$$
$$\sigma_r^*(b) = 0$$
$$\tau_{r\theta}^*(b) = 0$$

This problem corresponds to the one studied by Simmonds [29].

C. The Transverse Vibrations of a Flat Spinning Membrane Disk

In the case of a membrane disk the bending stiffness vanishes, so that D/t = 0. We will thus have

$$\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{r_0} w_r^*) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{\sigma_{\theta_0}}{r} w_\theta^*\right) = m\ddot{w}^* \tag{2.6.4}$$

which is the governing differential equation for the studies of several authors [5,6,7,10,11,12].

In this case the boundary conditions are different than for the case of the plate because of the vanishing of the bending stiffness, and hence the bending moment. At the hub radius only the transverse displacement can be set to zero, while at the outer edge only the requirement of the finiteness of the deflection can be imposed. Hence we will have

$$w(a) = 0$$
$$w(b) = \text{FINITE}$$

D. The Vibrations of a Spinning Shallow Spherical Membrane Shell

In the membrane shell theory, the bending stiffness vanishes so that Eqs. (2.5.9), (2.5.10), and (2.5.11) become

$$\frac{\partial}{\partial r}(r\sigma_r^*) - \sigma_\theta^* + \frac{\partial \tau_{r\theta}^*}{\partial \theta} = mr\left[\ddot{u}^* - 2\omega\dot{v}^* - \left(u^* + w^*\frac{r}{R}\right)\omega^2\right] \tag{2.6.5}$$

$$\frac{\partial}{\partial r}(r\tau_{r\theta}^*) + \frac{\partial \sigma_\theta^*}{\partial \theta} + \tau_{r\theta}^* = mr\left[\ddot{v}^* + 2\omega\left(\dot{u}^* + \dot{w}^*\frac{r}{R}\right) - v^*\omega^2\right] \tag{2.6.6}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left\{r\sigma_r^* w_{0r} + r\sigma_{r_0} w_r^*\right\} + \frac{1}{r}\frac{\partial}{\partial \theta}\left\{\frac{1}{r}\sigma_{\theta_0} w_\theta^* + w_{0r}\tau_{r\theta}^*\right\}$$

$$+\frac{1}{r}(\sigma_r^* + \sigma_\theta^*) = m\left[\ddot{w}^* - 2\omega\dot{v}^*\frac{r}{R} - \left(u^* + w^*\frac{r}{R}\right)\omega^2\frac{r}{R}\right] \tag{2.6.7}$$

These equations can be shown to be equivalent to those derived by Johnson [8] and solved for some special cases. This equivalence can be established by noting that Johnson utilizes displacements which are axial, radial, and tangential, whereas the present analysis uses normal, meridional, and tangential displacements. Within the shallow shell approximation the relationship between the two displacement systems is

$$\bar{u} = u + w\frac{r}{R}$$

$$\bar{v} = v$$

$$\bar{w} = w - u\frac{r}{R} \approx w \tag{2.6.8}$$

Furthermore, to the same order of approximation, the stresses σ_r and $\tau_{r\theta}$ are considered to be representative of either the mid-surface or "horizontal" stresses.

It can be seen that Eq. (2.6.6) becomes

$$\frac{\partial}{\partial r}\left(r\tau_{r\theta}^*\right)+\frac{\partial\sigma_\theta^*}{\partial\theta}+\tau_{r\theta}^* = mr\left[\ddot{\bar{v}}^*+2\omega\dot{\bar{u}}^*-\bar{v}^*\omega^2\right]$$

(2.6.9)

Equation (2.6.5) can be used to show that

$$\frac{\sigma_r^*+\sigma_\theta^*}{R}=\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r^2}{R}\sigma_r^*\right)+\frac{1}{R}\frac{\partial\tau_{r\theta}^*}{\partial\theta}-m\frac{r}{R}\left[\ddot{u}^*-2\omega\dot{v}^*-\left(u^*+w^*\frac{r}{R}\right)\omega^2\right]$$

(2.6.10)

By using Eq. (2.6.9), Eq. (2.6.7) can be written

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(-\frac{r}{R}+w_{0_r}\right)\sigma_r^*+r\sigma_{r_0}w_r^*\right]+\frac{1}{r}\frac{\partial}{\partial\theta}\left[\left(-\frac{r}{R}+w_{0_r}\right)\tau_{r\theta}^*+\frac{1}{r}\sigma_{\theta_0}w_\theta^*\right]$$
$$= m\left[\ddot{w}^*-\ddot{u}^*\frac{r}{R}\right]$$

(2.6.11)

The shallow shell relations of Eqs. (2.6.8) can be employed to yield

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(-\frac{r}{R}+\bar{w}_{0_r}\right)\sigma_r^*+r\sigma_{r_0}\bar{w}_r^*\right]+\frac{1}{r}\frac{\partial}{\partial\theta}\left[\left(-\frac{r}{R}+\bar{w}_{0_r}\right)\tau_{r\theta}^*+\frac{1}{r}\sigma_{\theta_0}\bar{w}_\theta^*\right]$$
$$= m\ddot{\bar{w}}^*$$

(2.6.12)

An equation for dynamic equilibrium in the horizontal direction is obtained by resolving Eqs. (2.6.5) and (2.6.7) in the horizontal direction and utilizing the shallow shell approximation that σ_θ and $\tau_{r\theta}$ are taken as the horizontal stresses. Following this procedure we obtain.

$$\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_r^*) - \frac{\sigma_\theta^*}{r} + \frac{1}{r}\frac{\partial \tau_{r\theta}^*}{\partial \theta} = m\Big[\Big(\ddot{u}^* + \ddot{w}^*\frac{r}{R}\Big)$$

$$- 2\omega\,\dot{v}^* - \Big(u^* + w^*\frac{r}{R}\Big)\omega^2\Big] \qquad (2.6.13)$$

where it has been assumed that

$$\frac{r}{R} \ll 1$$

Again, employing Eqs. (2.6.8) we obtain

$$\frac{\partial}{\partial r}(r\sigma_r^*) - \sigma_\theta^* + \frac{\partial \tau_{r\theta}^*}{\partial \theta} = mr\Big[\ddot{\bar{u}}^* - 2\omega\,\dot{\bar{v}}^* - \bar{u}^*\omega^2\Big]$$

(2.6.14)

In summary, for the case of displacements in the axial, radial, and tangential directions, we have, in the membrane case

$$\frac{\partial}{\partial r}(r\sigma_r^*) - \sigma_\theta^* + \frac{\partial \tau_{r\theta}^*}{\partial \theta} = mr\Big[\ddot{\bar{u}} - 2\omega\,\dot{\bar{v}}^* - \bar{u}^*\omega^2\Big]$$

(2.6.15)

$$\frac{\partial}{\partial r}(r\tau_{r\theta}^*) + \frac{\partial \sigma_\theta^*}{\partial \theta} + \tau_{r\theta}^* = mr\left[\ddot{v}^* + 2\omega\dot{u}^* - v^*\omega^2\right]$$

(2.6.16)

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(-\frac{r}{R} + \bar{w}_{0r}\right)\sigma_r^* + r\sigma_{r_0}\bar{w}_r^*\right] + \frac{1}{r}\frac{\partial}{\partial \theta}\left[\left(-\frac{r}{R} + \bar{w}_{0r}\right)\tau_{r\theta}^* + \frac{1}{r}\sigma_{\theta_0}\bar{w}_\theta^*\right] = m\ddot{\bar{w}}^*$$

(2.6.17)

Equations (2.6.15), (2.6.16), and (2.6.17) are identical to Johnson's equations specialized to the case of a spherical shell.

To supplement Eqs. (2.6.15), (2.6.16) and (2.6.17) it is necessary to give the mid-surface strain-displacement relations for the perturbation quantities. We previously have stated these relations for the axis system which is meridional, tangential, and normal. They were

$$\epsilon_r^* = u_r^* + \frac{w^*}{R} + w_{0r}w_r^* + \frac{1}{2}w_r^{*2}$$

$$\epsilon_\theta^* = \frac{v_\theta^*}{r} + \frac{u^*}{r} + \frac{w^*}{R} + \frac{1}{2}\left(\frac{w_\theta^*}{r}\right)^2$$

$$\gamma_{r\theta}^* = \frac{u_\theta^*}{r} + v_r^* - \frac{v^*}{r} + \left(\frac{w_{0r} + w_r^*}{r}\right)w_\theta^*$$

If we make the transition of co-ordinates given by

Eqs. (2.6.3) these become in the radial, tangential, and axial directions

$$\bar{\epsilon}_r^* = \bar{u}_r^* + \left(\bar{w}_{0r} - \frac{r}{R} \right) \bar{w}_r^* + \frac{1}{2} \bar{w}_r^{*2}$$

$$\bar{\epsilon}_\theta^* = \frac{\bar{v}_\theta^*}{r} + \frac{\bar{u}^*}{r} + \frac{1}{2} \left(\frac{\bar{w}_\theta^*}{r} \right)^2$$

$$\bar{\gamma}_{r\theta}^* = \frac{\bar{u}_\theta^*}{r} + \bar{v}_r^* - \frac{\bar{v}^*}{r} - \left(\frac{1}{R} - \frac{\bar{w}_{0r}}{r} \right) w_\theta^* + \frac{w_r^* w_\theta^*}{r}$$

(2.6.18)

In the case that bending is considered we can obtain a set of equations of motion for the radial, tangential, and axial axis system by including the bending term in Eq. (2.6.17) since within the scope of shallow shell theory it is unaltered. Hence, in addition to Eqs. (2.6.15) and (2.6.16) we will have

$$\frac{D}{t} \nabla^2 \nabla^2 \bar{w}^* - \frac{1}{r} \frac{\partial}{\partial r} \left\{ r \left(-\frac{r}{R} + \bar{w}_{0r} \right) \sigma_r^* + r \sigma_{r_0} \bar{w}_r^* \right\}$$
$$- \frac{1}{r} \frac{\partial}{\partial \theta} \left\{ \left(-\frac{r}{R} + \bar{w}_{0r} \right) \tau_{r\theta}^* + \frac{1}{r} \sigma_{\theta_0} \bar{w}_\theta^* \right\} = - m \ddot{\bar{w}}^*$$

(2.6.19)

The vibration equations in the form of Eqs. (2.6.15), (2.6.16), and (2.6.19) have the advantage that the orientation of the axis system is not tied to the initial shell shape.

2.7 Equations for Computation of Bending Moments

It has already been noted that the vanishing of the bending moment M_{rr} arises as one of the natural boundary conditions at the free edge of the spinning shell. However, it has been felt appropriate to give here expressions for the bending and twisting moments in the shell since they do not arise naturally in the course of variational development.

By appropriate integrations of the strain displacement relations of Eqs. (2.5.1) we obtain the moment resultants (moment per unit length)

$$M_{rr} = -D\left\{ w_{,rr} + \nu\left(\frac{1}{r}w_{,r} + \frac{1}{r^2}w_{,\theta\theta}\right)\right\}$$

$$M_{\theta\theta} = -D\left\{\frac{1}{r}w_{,r} + \frac{1}{r^2}w_{,\theta\theta} + \nu w_{,rr}\right\}$$

$$M_{r\theta} = -(1-\nu)D\frac{\partial}{\partial r}\left(\frac{1}{r}w_{,\theta}\right) \tag{2.7.1}$$

In the case of axisymmetric deformations these reduce to

$$M_{rr} = -D\left\{ w_{rr} + \frac{\nu}{r} w_r \right\}$$

$$M_{\theta\theta} = -D\left\{ \frac{w_r}{r} + \nu\, w_{rr} \right\}$$

$$M_{r\theta} = 0$$

(2.7.2)

CHAPTER 3

THE TRANSVERSE VIBRATIONS OF A SPINNING MEMBRANE DISK WITH A FULLY CLAMPED HUB

3.1 Introduction

As pointed out in Chapter 2, the general equations for the free vibrations of spinning shallow spherical shells contain as a special case, when the shell curvature is zero, the equations of motion for the vibrations of a flat membrane disk. It is found that the transverse vibrations are uncoupled from the in-plane vibrations. The case of in-plane vibrations was studied in detail by Simmonds [29]. Lamb and Southwell [5], Southwell [6], Simmonds [7, 11], Johnson [8], Bulkeley and Savage [10], and Eversman [12], have investigated the transverse vibrations of spinning membrane disks for several hub configurations and symmetry conditions. None of these authors has presented results for the case of general vibrations of a disk which has a hub configuration of significance for applications, as discussed in Chapter 1. It is the purpose of the next two chapters to study the transverse vibrations of spinning membrane disks which have hubs which provide various degrees of central clamping. Two cases are studied in detail. The case to be considered in this chapter corresponds to the case of a fully clamped, or built in, hub which constrains both vertical and radial displacements of the disk. The case to be considered in Chapter 4 is that of a "loosely clamped" hub which prevents vertical displacement but allows radial displacement. These two hub configurations are the extreme cases of physically significant hub possibilities. The intermediate hub configurations have been studied for axisymmetric vibrations by Bulkeley and Savage [10] and their results are used to formulate the equations for these hub conditions for the asymmetric case at the end of Chapter 4. The hub configurations are shown in Figure 5.

3.2 The Equilibrium Stress Distribution

The hub-disk configuration to be analyzed is shown in Figure 6. The differential equation which governs the equilibrium stress distribution in a spinning disk was obtained as a special case of the spinning shallow spherical shell. It is given by Eq. (2.4.2) and is repeated here for convenience

$$\frac{d}{dr}(r\sigma_r) - \sigma_\theta + m\omega^2 r^2 = 0 \tag{3.2.1}$$

In the case of the fully clamped hub the boundary conditions to be satisfied are

$$u(a) = 0$$

$$\sigma_r(b) = 0$$

where $r = a$ is the hub radius and $r = b$ is the disk radius.

Solutions for the equilibrium displacement and stress distributions are in the form [28]

$$u = C_1 r + \frac{C_2}{r} - \frac{(1-\nu^2)}{8E} m\omega^2 r^3 \tag{3.2.2}$$

$$\sigma_r = \frac{E}{1-\nu^2}\left\{(1+\nu)C_1 + (1-\nu)\frac{C_2}{r^2} - \frac{(1+3\nu)(1-\nu^2)}{8E} m\omega^2 r^2\right\} \tag{3.2.3}$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left\{(1+\nu)C_1 - (1-\nu)\frac{C_2}{r^2} - \frac{(3+\nu)(1-\nu^2)}{8E} m\omega^2 r^2\right\} \tag{3.2.4}$$

The boundary conditions lead to two simultaneous equations for the constants C_1 and C_2

$$C_1 + \frac{C_2}{a^2} = \frac{(1-\nu^2)}{8E} m\omega^2 a^2$$

(3.2.5)

$$(1+\nu)C_1 - (1-\nu)\frac{C_2}{b^2} = \frac{(3+\nu)(1-\nu^2)}{8E} m\omega^2 b^2$$

(3.2.6)

By solving Eqs. (3.2.5) and (3.2.6) we find for C_1 and C_2

$$C_1 = \frac{(3+\nu)m\omega^2}{8E}(1-\nu)\left\{b^2 + \frac{(1-\nu)}{(3+\nu)}a^2\left[\frac{(1+\nu)a^2-(3+\nu)b^2}{(1+\nu)b^2+(1-\nu)a^2}\right]\right\}$$

(3.2.7)

$$C_2 = \frac{(1-\nu)m\omega^2}{8E}a^2b^2\left[\frac{(1+\nu)a^2-(3+\nu)b^2}{(1+\nu)b^2+(1-\nu)a^2}\right]$$

(3.2.8)

For the equilibrium stresses we thus have

$$\sigma_r = \frac{\mu_2}{r^2}(b^2-r^2)\left(r^2 + \frac{\mu_1}{b^2\mu_2}\right)$$

(3.2.9)

$$\sigma_\theta = \frac{\mu_2}{r^2}\left\{\left(b^2 - \frac{\mu_1}{b^2\mu_2}\right) r^2 - \frac{\mu_1}{\mu_2} - \frac{\mu_3}{\mu_2} r^4\right\} \tag{3.2.10}$$

where

$$\mu_1 = \frac{(1-\nu)}{8} m\omega^2 a^2 b^2 \left[\frac{(3+\nu) b^2 - (1-\nu) a^2}{(1+\nu) b^2 + (1-\nu) a^2}\right] \tag{3.2.11}$$

$$\mu_2 = \frac{3+\nu}{8} m\omega^2 \tag{3.2.12}$$

$$\mu_3 = \frac{1+3\nu}{8} m\omega^2 \tag{3.2.13}$$

3.3 The Governing Differential Equation for Free Vibrations

The partial differential equation which governs the transverse vibrations of a spinning membrane disk is given by Eq. (2.6.4) and is repeated here

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\sigma_r \frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\sigma_\theta \frac{\partial w}{\partial \theta}\right) - m\frac{\partial^2 w}{\partial t^2} = 0 \tag{3.3.1}$$

Where σ_r and σ_θ are the equilibrium stress distributions given by Eqs. (3.2.9) and (3.2.10). The boundary conditions to be satisfied for the fully clamped hub are

$$w(a) = 0$$

$$w(b) = \text{FINITE}$$

Hence we seek solutions of Eq. (3.3.1) which vanish at the hub and are finite at the outer edge.

We assume a solution of the form

$$w(r, \theta, t) = W(r) \sin s\theta \sin pt$$

which, when substituted in Eq. (3.3.1), yields the following ordinary differential equation for $W(r)$

$$\frac{1}{r}\frac{d}{dr}\left(r\sigma_r\frac{dW}{dr}\right) + \left(\lambda^2 - \frac{s^2\sigma_\theta}{r^2}\right) W = 0 \tag{3.3.2}$$

where

$$p^2 = \frac{\lambda^2}{m}$$

It is convenient to partially non-dimensionalize Eq. (3.3.2) by introducing the non-dimensional radius ρ defined by

$$r = \rho b$$

If we introduce this into Eqs. (3.2.9) and (3.2.10) for the stresses we obtain

$$\sigma_r = \frac{b^2 \mu_2}{\rho^2} (1-\rho^2)(\rho^2+\delta^2) \tag{3.3.3}$$

$$\sigma_\theta = \frac{b^2 \mu_2}{\rho^2} \left\{ (1-\delta^2)\rho^2 - \delta^2 - \frac{1+3\nu}{3+\nu} \rho^4 \right\} \tag{3.3.4}$$

where

$$\delta^2 = \frac{\mu_1}{b^4 \mu_2} = \frac{1-\nu}{3+\nu} \left(\frac{a}{b}\right)^4 \left[\frac{(3+\nu)b^2 - (1+\nu)a^2}{(1+\nu)b^2 + (1-\nu)a^2} \right] \tag{3.3.5}$$

With the introduction of the non-dimensional radius and Eqs. (3.3.3) and (3.3.4) into Eq. (3.3.2) there results

$$\frac{1}{\rho}\frac{d}{d\rho}\left[\frac{1}{\rho}(1-\rho^2)(\rho^2+\delta^2)\frac{dW}{d\rho}\right] + \left\{ \left(\frac{\lambda^2}{\mu_2} + \frac{1+3\nu}{3+\nu} s^2\right) - s^4\left[(1-\delta^2)\frac{1}{\rho^2} - \frac{\delta^2}{\rho^4}\right] \right\} W = 0 \tag{3.3.6}$$

Further simplification can be obtained by introducing the change of variable

$$x = \rho^2$$

and the definition

$$\mu^2 = \frac{\lambda^2}{\mu_2} + \frac{1+3\nu}{3+\nu} s^2 = \frac{8}{3+\nu}\left[\left(\frac{\Omega}{\omega}\right)^2 + \frac{(1+3\nu)}{8} s^2\right]$$

to obtain

$$x(x-1)(x+\delta^2)\frac{d^2W}{dx^2} + \left[2x^2-(1-\delta^2)x\right]\frac{dW}{dx} + \frac{1}{4}\left\{\left[(1-\delta^2)-\frac{\delta^2}{x}\right]s^2 - \mu^2 x\right\}W = 0$$

(3.3.7)

In this form the boundary conditions to be satisfied are

$$W\left[\left(\frac{a}{b}\right)^2\right] = 0$$

$$W(1) = \text{FINITE}$$

(3.3.8)

3.4 Solutions of the Differential Equation

The basic differential equation for the radial dependence of the vibration mode shapes is given by Eq. (3.3.7) and supplemented by Eqs. (3.3.8) for the boundary conditions. Equation (3.3.7) is of the Fuchsian type with four regular singular points at x=0, x=1, $x=-\delta^2$ and $x=\infty$. It can be reduced by an appropriate transformation of the dependent variable to Heun's equation [32]. In the particular case at hand, direct reduction to Heun's equation, at least by conventional means, does not appear practical since it involves a transformation of the form

$$W(x) = x^k y(x)$$

in which k is complex for vibration modes with more than one nodal diameter. Another transformation which will lead to Heun's equation has been used by Eversman [12] in the case of the annular membrane with free edges. In this approach the transformation

$$\zeta = \frac{1}{x}$$

is made and then a transformation of the form

$$W(\zeta) = \zeta^k y(\zeta)$$

This reduction is successful in that Heun's equation can be obtained with real values of k. However, the transformed geometry of the singular points must be considered. After the change of independent variables there will be singular points at $\zeta = 0$, $\zeta = 1$, $\zeta = -1/\delta^2$, and $\zeta = \infty$. The region of physical interest lies between $\zeta = 1$ and $\zeta = (b/a)^2$. Since the distance between $\zeta = (b/a)^2$ and $\zeta = 1$ is less than 1 only if $(b/a)^2 < 2$, and since series solutions about $\zeta = 1$ can be guaranteed to converge only for $0 < \zeta < 2$, only certain disk geometries can be conveniently treated.

Because of these observations and restrictions, no further reduction of the basic differential equation is made.

Solutions to Eq. (3.3.8) can be obtained in the form of a power series expansion about one of the singular points [33]. The singular point at the free edge is the only one about which an expansion can be made which can be proved to converge for the entire physical region for any disk geometry. Hence it is appropriate to shift this point to the origin with the transformation

$$\zeta = -\alpha^2(x-1) \tag{3.4.1}$$

where

$$\alpha^2 = \frac{1}{1-(a/b)^2} \tag{3.4.2}$$

With this transformation the singular point geometry changes as follows:

$$x = 1 \implies \zeta = 0$$

$$x = -\delta^2 \implies \zeta = \alpha^2(1+\delta^2)$$

$$x = 0 \implies \zeta = \alpha^2$$

The hub radius, originally at $\chi = (a/b)^2$, transforms to $\zeta = 1$. The differential equation transforms to

$$\zeta\left[\zeta - \alpha^2(1+\delta^2)\right](\zeta-\alpha^2)^2 \frac{d^2W}{d\zeta^2} + (\zeta-\alpha^2)^2\left\{\zeta + \left[\zeta - \alpha^2(1+\delta^2)\right]\right\}\frac{dW}{d\zeta}$$

$$+\frac{1}{4}\left\{\left[-\alpha^2(1-\delta^2)(\zeta-\alpha^2) - \alpha^4\delta^2\right]\zeta^2 - \mu^2(\zeta-\alpha^2)^2\right\}W = 0 \tag{3.4.3}$$

and the boundary conditions become

$$W(0) = \text{FINITE}$$

$$W(1) = 0 \tag{3.4.4}$$

Solutions to Eq. (3.4.3) are sought in the form

$$W(\zeta) = \sum_{n=0}^{\infty} C_n \zeta^{n+\rho} \tag{3.4.5}$$

If we introduce the definitions

$$\alpha_n = (n+\rho)(n+\rho-1) + 2(n+\rho) - \frac{\mu^2}{4}$$

$$\beta_n = -\alpha^2\left\{(3+\delta^2)(n+\rho)(n+\rho-1) + (5+\delta^2)(n+\rho) - \frac{1}{4}\left[2\mu^2 - s^2(1-\delta^2)\right]\right\}$$

$$\gamma_n = \alpha^4\left\{(3+2\delta^2)(n+\rho)(n+\rho-1) + (4+2\delta^2)(n+\rho) + \frac{1}{4}\left[(1-2\delta^2)s^2 - \mu^2\right]\right\}$$

$$\delta_n = -\alpha^6(1+\delta^2)(n+\rho)^2$$

it is found that the unknown coefficients, C_n, are defined by the recurrence relations

$$\delta_o C_o = 0$$

$$\gamma_o C_o + \delta_1 C_1 = 0$$

$$\beta_o C_o + \gamma_1 C_1 + \delta_2 C_2 = 0$$

$$\alpha_{n-2} C_{n-2} + \beta_{n-1} C_{n-1} + \gamma_n C_n + \delta_{n+1} C_{n+1} = 0; \quad n \geq 2$$

The first of these relations is the indicial equation. For arbitrary C_o it follows that

$$\delta_o = 0$$

or

$$-\alpha^6(1+\delta^2)\rho^2 = 0$$

This yields the two indicial exponents of the solution

$$\rho_1 = 0$$

$$\rho_2 = 0$$

With $\rho = 0$ there exists a solution

$$W = \sum_{n=0}^{\infty} C_n \zeta^n \tag{3.4.6}$$

with the C_n defined by

$$\gamma_0 C_0 + \delta_1 C_1 = 0$$

$$\beta_0 C_0 + \gamma_1 C_1 + \delta_2 C_2 = 0$$

$$\alpha_{n-2} C_{n-2} + \beta_{n-1} C_{n-1} + \gamma_n C_n + \delta_{n+1} C_{n+1} = 0 \; ; \quad n \geq 2 \tag{3.4.7}$$

where

$$\alpha_n = n(n+1) - \frac{\mu^2}{4}$$

$$\beta_n = -\alpha^2\left\{(3+\delta^2)\,n(n-1) + (5+\delta^2)\,n - \frac{1}{4}\left[2\mu^2 - s^2(1-\delta^2)\right]\right\}$$

$$\gamma_n = \alpha^4\left\{(3+2\delta^2)\,n(n-1) + (4+2\delta^2)\,n + \frac{1}{4}\left[(1-2\delta^2)s^2 - \mu^2\right]\right\}$$

$$\delta_n = -\alpha^6(1+\delta^2)\,n^2 \qquad (3.4.8)$$

Since the characteristic exponents of the solution are repeated, only one solution of the above form exists. The second independent solution is known to have a logarithmic singularity at $\zeta = 0$ [33] and is discarded to satisfy the boundary condition at that point.

3.5 The Eigenvalue Problem

The boundary condition of finiteness at $\zeta = 0$ is satisfied by the solution given by Eqs. (3.4.6), (3.4.7) and (3.4.8). The eigenvalue problem is that of finding values of the parameter μ^2, and hence the natural frequency p, for which a solution of this form will satisfy the boundary condition $W(1) = 0$ with C_0 not zero. By noting that α_n, β_n and γ_n are functions of μ^2 we conclude that

$$C_n = C_n(\mu^2)$$

Hence the eigenvalue equation is

$$W(1) = \sum_{n=0}^{\infty}{}' C_n(\mu^2) = 0 \tag{3.5.1}$$

and we seek to find the zeroes of the transcendental function

$$F(\mu^2) = \sum_{n=0}^{\infty}{}' C_n(\mu^2) \tag{3.5.2}$$

3.6 Numerical Evaluation of the Eigenvalues

A digital computer program has been written to find the roots of the eigenvalue equation

$$F(\mu^2) = 0 \tag{3.6.1}$$

The method employed consists of evaluating $F(\mu^2)$ for a sequence of values of μ^2 until a change in sign is noted. When the change of sign occurs, and the presence of a root is thus indicated, an iteration procedure based on the method of secants [34] is employed to obtain the eigenvalue. This procedure is carried out for a given number of nodal diameters s and for as many values of δ in $0 \leq (a/b) \leq 1$ as specified. For specific values of s and $(\frac{a}{b})$ any number of roots can be calculated, beginning with the smallest and proceeding in ascending order.

After a particular eigenvalue is obtained, the eigenfunction can be evaluated, by program option, by evaluating the solution for the known eigenvalue μ^2, at as many points in the interval $0 \leq \zeta \leq 1$, or $a \leq r \leq b$, as necessary for definition.

The above numerical technique was programmed for the IBM 1620 Digital Computer at Wichita State University. It was found that accuracy and convergence of the iteration scheme could not be maintained without the use of double precision arithmetic. This, coupled with the inherent complexity of the evaluation and iteration calculations, led to computational times per root which were impractical for production computation on the 1620. Through the kind cooperation of the IBM representative access was made available to an IBM 360, Model 65, as well as a Model 75, for the purpose of completing the numerical work. Computation times on these machines were only a few seconds per root.

3.7 The Special Case of Symmetric Vibrations

In the special case of symmetric vibrations the parameter $S = 0$ and Eq. (3.3.7) reduces to

$$(x-1)(x+\delta^2)\frac{d^2W}{dx^2} + \left[2x-(1-\delta^2)\right]\frac{dW}{dx} - \frac{\mu^2}{4}W = 0 \tag{3.7.1}$$

where

$$\mu^2 = \frac{8}{3+\nu}\left(\frac{p}{\omega}\right)^2 \tag{3.7.2}$$

If the transformation

$$\zeta = \frac{1-x}{1+\delta^2}$$

is introduced into Eq. (3.7.1) there results the differential equation

$$\zeta(\zeta-1)\frac{d^2W}{d\zeta^2}+\left[2\zeta-1\right]\frac{dW}{d\zeta}-\frac{2}{3+\nu}\left(\frac{p}{\omega}\right)^2 W=0$$

(3.7.3)

with the boundary conditions

$$W(0) = \text{FINITE}$$

$$W\left[\frac{1-(a/b)^2}{1+\delta^2}\right] = 0$$

(3.7.4)

The mathematical problem specified by Eqs. (3.7.3) and (3.7.4) was studied by Simmonds [11] in his investigations of this case.

It is found in this case that there exists one solution of the form

$$W(\zeta) = \sum_{n=0}^{\infty} C_n \zeta^n$$

(3.7.5)

where the C_n are defined by

$$C_{n+1} = \frac{n(n+1) - \mu^2/4 \, C_n}{(n+1)^2}$$

(3.7.6)

The second solution is logarithmically singular at $\zeta = 0$ and is excluded to satisfy the boundary condition at that point.

The eigenvalue problem is that of determining μ^2 such that

$$G(\mu^2) = \sum_{n=0}^{\infty} C_n(\mu^2)\left[\frac{1-(a/b)^2}{1+\delta^2}\right]^n = 0$$

(3.7.7)

so that the second boundary condition is satisfied. As a means of verifying the results of the general eigenvalue problem specified by Eq. (3.6.1), a digital computer program for the special case of symmetric vibrations was developed. This program utilized an iteration technique similar to the one discussed in Section (3.6) and was based on the eigenvalue problem specified by Eq. (3.7.7). The results of these calculations showed good correlation with Simmonds' results and agreed perfectly with the results for the case $s = 0$ from the general problem.

3.8 The Special Case With One Nodal Diameter

In order to further verify the analysis it was decided to also treat independently the case of vibrations with one nodal diameter. In this case we have $s = 1$ and the governing differential equation for the radial mode shapes is

$$x(x-1)(x+\delta^2)\frac{d^2W}{dx^2} + \left[2x^2 - (1-\delta^2)x\right]\frac{dW}{dx} + \frac{1}{4}\left\{\left[(1-\delta^2) - \frac{\delta^2}{x}\right] - \mu^2 x\right\}W = 0 \tag{3.8.1}$$

where

$$\mu^2 = \frac{8}{3+\nu}\left[\left(\frac{\Omega}{\omega}\right)^2 + \frac{1+3\nu}{8}\right] \tag{3.8.2}$$

In this case it is convenient to simplify the differential equation by introduction of the change of variable

$$W(x) = x^{1/2} Y(x) \tag{3.8.3}$$

With this substitution Eq. (3.8.1) becomes

$$x(x-1)(x+\delta^2)\frac{d^2Y}{dx^2} + \left\{(x-1)(x+\delta^2) + x(x-1) + x(x+\delta^2)\right\}\frac{dY}{dx} + \frac{1}{4}(3-\mu^2)xY = 0 \tag{3.8.4}$$

Equation (3.8.4) is a form of Heun's Differential Equation [35].

As in previous cases, it is convenient to expand the solution about the point $x = 1$. This point is shifted to the origin by the transformation

$$\zeta = \frac{x-1}{(a/b)^2 - 1} = -\alpha^2(x-1) \tag{3.8.5}$$

The differential equation becomes

$$\zeta(\zeta-\alpha^2)[\zeta-\alpha^2(1+\delta^2)]\,\frac{d^2Y}{d\zeta^2}$$

$$+\left\{\zeta[\zeta-\alpha^2(1+\delta^2)]+\zeta(\zeta-\alpha^2)+(\zeta-\alpha^2)[\zeta-\alpha^2(1+\delta^2)]\right\}$$

$$+\frac{1}{4}(3-\mu^2)(\zeta-\alpha^2)Y = 0 \tag{3.8.6}$$

and the corresponding boundary conditions, deduced from Eqs. (3.4.4), are

$$Y(0) = \text{FINITE}$$

$$Y(1) = 0 \tag{3.8.7}$$

One solution to Eq. (3.8.6) exists in the form

$$y(\zeta) = \sum_{n=0}^{\infty} C_n \zeta^n \tag{3.8.8}$$

where the recurrence relation for the C_n is

$$\gamma_1 C_1 + \beta_0 C_0 = 0$$

$$\alpha_{n-1} C_{n-1} + \beta_n C_n + \gamma_{n+1} C_{n+1} = 0 \tag{3.8.9}$$

and

$$\alpha_n = n(n+2) + \frac{1}{4}(3-\mu^2)$$

$$\beta_n = -\alpha^2\left[(2+\delta^2)\,n(n-1) + (4+2\delta^2)\,n + \frac{1}{4}(3-\mu^2)\right]$$

$$\gamma_n = \alpha^4(1+\delta^2)\,n^2 \tag{3.8.10}$$

A second solution exists which is logarithmically singular at $\zeta = 0$. The boundary condition of finiteness at the origin requires that this solution be discarded.

At this point it is interesting to note the simplification introduced by the transformation to Heun's Equation. The recurrence relation given by Eqs. (3.8.9) contains only three terms, but it would have contained four terms if the transformation had not been accomplished. A similar result would have been observed in the general case, but as previously noted, other difficulties precluded the transformation.

In this case the eigenvalue problem is one of determining μ^2 such that

$$H(\mu^2) = \sum_{n=0}^{\infty} C_n(\mu^2) = 0 \tag{3.8.11}$$

The roots of this transcendental equation were found by a method similar to the ones employed for the symmetric and general cases. The results were compared with those obtained from the general program for the case $S=1$ and were found to be in complete agreement.

3.9 Results

The numerical procedures discussed in the previous section were used to calculate the first four eigenvalues, $n=0$ to $n=3$, for cases of from zero to three nodal diameters, $S=0$ to $S=3$. The results of these calculations are plotted in Fig. 7 as a function of the ratio of the hub radius to the disk radius.

The general trend is for the increase of μ^2, and hence the vibration frequency, with an increase in the hub size. This is reasonable from a physical viewpoint since increasing the hub radius would appear to stiffen the system.

An exception to this observation occurs for the higher modes corresponding to a given number of nodal circles n.

It is observed, for example, that for $n=0$ and $s=2$ and 3 that there is an initial reduction of μ^2 with increasing δ. It is anticipated that this trend will appear whenever the number of nodal diameters is substantially larger than the number of nodal circles. This trend can be explained in terms of the dependence of the local disk stiffness on the nodal geometry. The equilibrium stresses in the disk are given by

$$\sigma_r = \frac{b^2\mu_z}{\rho^2}(1-\rho^2)(\rho^2+\delta^2) \tag{3.9.1}$$

$$\sigma_\theta = \frac{b^2\mu_z}{\rho^2}\left\{(1-\delta^2)\rho^2 - \delta^2 - \frac{1+3\nu}{3+\nu}\rho^4\right\} \tag{3.9.2}$$

where

$$\delta^2 = \frac{1-\nu}{3+\nu}\left(\frac{a}{b}\right)^2\left[\frac{(3+\nu)b^2-(1+\nu)a^2}{(1+\nu)b^2+(1-\nu)a^2}\right] \tag{3.9.3}$$

We note that

$$\frac{d\sigma_r}{d\delta^2} = \frac{b^2\mu_2}{\rho^2}(1-\rho^2) > 0$$

(3.9.4)

$$\frac{d\sigma_\theta}{d\delta^2} = -\frac{b^2\mu_2}{\rho^2}(\rho^2+1) < 0$$

(3.9.5)

so that for $0 \leq \rho \leq 1$ the local radial stress increases with increasing δ while the tangential stress decreases with increasing δ. Since δ depends directly on $(\frac{a}{b})^2$ the same can be said about increasing or decreasing the ratio of the hub radius to the disk radius.

If there are substantially more nodal diameters than nodal circles the local membrane stiffness will depend predominantly more on the tangential stress than on the radial stress for small $(\frac{a}{b})$ and will tend to decrease with increasing $(\frac{a}{b})$ up to a limiting value of $(\frac{a}{b})$. It would be expected that the more the number of nodal diameters exceeds the number of nodal circles, the more pronounced this effect would be and the larger would be the range of $(\frac{a}{b})$ over which it is observed.

If, on the other hand, the nodal geometry tends to make σ_r predominant in governing the local stiffness, then the trend of increasing μ^2 with increasing $(\frac{a}{b})$ is expected.

A second interesting feature occurs for $(\frac{a}{b})$ near unity where for the the higher values of n there is little variation of μ^2 with s. This can be explained by physically reasoning that for large n and large $(\frac{a}{b})$ the stiffness is governed almost completely by the nodal circle geometry since they are very closely spaced while the nodal diameters are widely spaced in this region. For large s it would be expected that more variation with s would be seen.

CHAPTER 4

THE TRANSVERSE VIBRATIONS OF A SPINNING ANNULAR MEMBRANE DISK WITH A FRICTIONLESS HUB

4.1 Introduction

Treated in this chapter is the problem of the transverse vibrations of a spinning annular membrane disk with a frictionless or loosely clamped hub. By this it is meant that the hub does not restrain in-plane motions of the disk but prevents transverse displacements. From a physical standpoint this case is similar to the case discussed and solved by Simmonds [36], except here the disk is annular rather than solid, so that the presence of a hub is on a more physically realistic basis. The author was assisted by Mr. Thomas Gilley [37] in the following analysis.

4.2 The Equilibrium Stress Distribution

The differential equation and boundary conditions governing the equilibrium stress distribution in the case of a loosely clamped hub are

$$\frac{d}{dr}(r\sigma_r) - \sigma_\theta + m\omega^2 r^2 = 0$$

$$\sigma_r(a) = 0$$

$$\sigma_r(b) = 0 \qquad (4.2.1)$$

where a and b are the inner and outer annulus radii. Solutions to the differential equation are given by Eqs. (3.2.2), (3.2.3), and (3.2.4). By employing the two boundary conditions the two constants, C_1 and C_2, can be obtained and the solutions for the stresses are found to be [28]

$$\sigma_r = \frac{m\omega^2(3+\nu)}{8}\left(a^2+b^2-\frac{a^2b^2}{r^2}-r^2\right) \tag{4.2.2}$$

$$\sigma_\theta = \frac{m\omega^2(3+\nu)}{8}\left[a^2+b^2+\frac{a^2b^2}{r^2}-\frac{1+3\nu}{3+\nu}r^2\right] \tag{4.2.3}$$

In a partly non-dimensional form, with

$$r = \rho b \tag{4.2.4}$$

Eqs. (4.2.2) and (4.2.3) can be written

$$\sigma_r = m\omega^2 b\,\frac{3+\nu}{8}\,\frac{1}{\rho^2}(\rho^2-1)(\delta^2-\rho^2) \tag{4.2.5}$$

$$\sigma_\theta = m\omega^2 b^2 \frac{3+\nu}{8} \frac{1}{\rho^2}\left[(1+\delta^2)\rho^2 + \delta^2 - \frac{1+3\nu}{3+\nu}\rho^4\right] \qquad (4.2.6)$$

where

$$\delta = \frac{a}{b} \qquad (4.2.7)$$

is the annulus radius ratio. It is interesting to note the similarity between the functional form of the above stresses with those given by Eqs. (3.3.3) and (3.3.4), keeping in mind, however, the difference in the definition of δ .

4.3 The Governing Differential Equation for Free Vibrations

By substituting the stress distribution given by Eqs. (4.2.6) and (4.2.7) into the governing differential equation for transverse vibrations, given by Eq. (2.6.4), separating variables, and using the non-dimensional radius defined by Eq. (4.2.4), there is obtained for the radial mode shapes in the present case

$$\frac{1}{\rho}\frac{d}{d\rho}\left[\frac{1}{\rho}(\rho^2-1)(\delta^2-\rho^2)\frac{dW}{d\rho}\right]$$
$$+\left\{\left(\frac{\lambda^2}{\mu_2}+\frac{1+3\nu}{3+\nu}s^2\right)-s^2\left[(1+\delta^2)\frac{1}{\rho^2}+\frac{\delta^2}{\rho^2}\right]\right\}W = 0 \tag{4.3.1}$$

With the change of variable

$$x = \rho^2 \tag{4.3.2}$$

Eq. (4.3.1) becomes

$$x(x-1)(x-\delta^2)\frac{d^2W}{dx^2}+\left[2x^2-(1+\delta^2)x\right]\frac{dW}{dx}$$
$$+\frac{1}{4}\left\{\left[(1+\delta^2)+\frac{\delta^2}{x}\right]s^2-\mu^2 x\right\}W = 0 \tag{4.3.3}$$

where

$$\mu^2 = \frac{8}{3+\nu}\left[\left(\frac{\Omega}{\omega}\right)^2+\frac{1+3\nu}{8}s^2\right] \tag{4.3.4}$$

The boundary conditions imposed by the hub on the transverse vibrations of the disk require

$$W(c) = 0$$

$$W(b) = \text{FINITE}$$

where $r=c$ is the radius of the hub. In terms of the variable x these conditions become

$$W\left(\frac{c^2}{b^2}\right) = W(\epsilon^2) = 0 \tag{4.3.5}$$

$$W(1) = \text{FINITE} \tag{4.3.6}$$

where $\epsilon^2 = c^2/b^2$ is the square of the ratio of the hub radius to the disk radius.

4.4 Solutions of the Differential Equation

The observations of Section (3.4) concerning the difficulties of reducing the governing differential equation to Heun's equation are also pertinent for the present case. For this reason, Eq. (4.3.3) will be treated without further reduction.

An expansion in a power series about the singular point at $x=1$ is facilitated if we shift this point to the origin with the transformation

$$\zeta = -\alpha^2(x-1)$$

(4.4.1)

where

$$\alpha^2 = \frac{1}{1-(c/b)^2} = \frac{1}{1-\epsilon^2}$$

(4.4.2)

Accordingly, the singular point geometry shifts as follows:

$$x = 1 \implies \zeta = 0$$
$$x = \delta^2 \implies \zeta = \alpha^2(1-\delta^2)$$
$$x = 0 \implies \zeta = \alpha^2$$

The hub radius, originally at $x = \epsilon^2$, shifts to $\zeta = 1$. By introducing this transformation into the differential equation we obtain

$$\zeta[\zeta-\alpha^2(1-\delta^2)](\zeta-\alpha^2)^2\frac{d^2W}{d\zeta^2} + (\zeta-\alpha^2)^2\{\zeta+[\zeta-\alpha^2(1-\delta^2)]\}\frac{dW}{d\zeta}$$

$$+\frac{1}{4}\{[-\alpha^2(1+\delta^2)(\zeta-\alpha^2)+\alpha^4\delta^2]s^2 - \mu^2(\zeta-\alpha^2)^2\}W = 0$$

(4.4.3)

The boundary conditions become

$$W(0) = \text{FINITE}$$

$$W(1) = 0 \tag{4.4.4}$$

It is found that there exists one solution of the form

$$W = \sum_{n=0}^{\infty} C_n \zeta^n \tag{4.4.5}$$

where the C_n are defined by the recurrence relation

$$\gamma_0 C_0 + \delta_1 C_1 = 0$$

$$\beta_0 C_0 + \gamma_1 C_1 + \delta_2 C_2 = 0$$

$$\alpha_{n-2} C_{n-2} + \beta_{n-1} C_{n-1} + \gamma_n C_n + \delta_{n+1} C_{n+1} = 0 \; ; \quad n \geq 2 \tag{4.4.6}$$

The coefficients α_n, β_n, γ_n, δ_n are given by

$$\alpha_n = n(n+1) - \mu^2/4$$

$$\beta_n = -a^2 \left\{ (3-\delta^2) n(n-1) + (5-\delta^2) n - \tfrac{1}{4}\left[2\mu^2 - s^2(1+\delta^2) \right] \right\}$$

$$\gamma_n = a^4 \left\{ (3-2\delta^2) n(n-1) + (4-2\delta^2) n + \tfrac{1}{4}\left[(1+2\delta^2) s^2 - \mu^2 \right] \right\}$$

$$\delta_n = -a^6 (1-\delta^2) n^2 \tag{4.4.7}$$

A second independent solution of the form of Eq. (4.4.5) does not exist because the characteristic exponents of the solution are repeated. The second independent solution is logarithmically singular at $\zeta = 0$ and hence is excluded to satisfy the boundary condition of finiteness at that point.

4.5 The Eigenvalue Problem

The power series solution given by Eq. (4.4.5) and supplemented by Eqs. (4.4.6) and (4.4.7) satisfies the condition of finiteness at $\zeta = 0$. The boundary condition of no displacement at the hub, $\zeta = 1$, will be satisfied if

$$W(1) = \sum_{n=0}^{\infty} C_n(\mu^2) = 0 \tag{4.5.1}$$

Equation (4.5.1), with the $C_n(\mu^2)$ defined by the recurrence relation in Eqs. (4.4.6) and (4.4.7), constitutes a transcendental equation defining the values of μ^2 for which $W(1)$ vanishes with C_0 not zero. Hence the eigenvalue problem is that of finding the zeros of the function

$$F(\mu^2) = \sum_{n=0}^{\infty} C_n(\mu^2) \tag{4.5.2}$$

A numerical technique similar to the one described in Section (3.6) was used to numerically evaluate the roots of Eq. (4.5.1). The present case has the added complication that in addition to nodal geometry, defined by integer values of s, we must consider variations in both the ratio of the hub radius to the disk radius, ϵ , and the annulus radius ratio, δ .

4.6 The Special Cases of Symmetric Vibrations and Vibrations With One Nodal Diameter

As in the case of the fully clamped hub, there are two special cases which reduce the complexity of the governing differential equation and can be used independently of the general development for computational purposes or for verification of results computed from the general case.

The case of symmetric vibrations, with $s=0$, leads to the differential equation

$$x(x-1)(x-\delta^2)\frac{d^2W}{dx^2} + \left[2x-(1+\delta^2)\right]\frac{dW}{dx} - \frac{\mu^2}{4}W = 0 \tag{4.6.1}$$

where

$$\mu^2 = \frac{8}{3+\nu}\left(\frac{p}{\omega}\right)^2$$

By introducing the change of variable

$$\zeta = \frac{1-x}{1-\delta^2}$$

we obtain for the differential equation and boundary conditions

$$\zeta(\zeta-1)\frac{d^2W}{d\zeta^2} + \left[(\zeta-1)+\zeta\right]\frac{dW}{d\zeta} - \frac{\mu^2}{4}W = 0$$

(4.6.2)

$$W(0) = \text{FINITE}$$

$$W\left[\frac{1-\epsilon^2}{1-\delta^2}\right] = 1 \qquad (4.6.3)$$

It is found that the solution satisfying the condition of finiteness at the origin is of the form

$$W(\zeta) = \sum_{n=0}^{\infty} C_n \zeta^n$$

(4.6.4)

with

$$C_{n+1} = \frac{n(n+1) - \mu^2/4}{(n+1)^2}\, C_n$$

(4.6.5)

The eigenvalue problem is that of determining μ^2 such that

$$G(\mu^2) = \sum_{n=0}^{\infty} C_n(\mu^2)\left[\frac{1-\epsilon^2}{1-\delta^2}\right]^n = 0$$

(4.6.6)

In the case of one nodal diameter, the differential equation is

$$x(x-1)(x-\delta^2)\frac{d^2W}{dx^2} + \left[2x^2-(1+\delta^2)x\right]\frac{dW}{dx}$$

$$+\frac{1}{4}\left\{\left[(1-\delta^2)+\frac{\delta^2}{x}\right]-\mu^2 x\right\}W = 0 \qquad (4.6.7)$$

With the change of dependent variable

$$W = x^{1/2}\,y(x)$$

and the subsequent change of independent variable

$$\zeta = \frac{1-x}{1-\epsilon^2} = \alpha^2(1-x)$$

(4.6.8)

the differential equation becomes

$$\zeta(\zeta-\alpha^2)[\zeta-\alpha^2(1-\delta^2)]\frac{d^2Y}{d\zeta^2}$$

$$+\{\zeta[\zeta-\alpha^2(1-\delta^2)]+\zeta(\zeta-\alpha^2)+(\zeta-\alpha^2)[\zeta-\alpha^2(1-\delta^2)]\}\frac{dY}{d\zeta}$$

$$+\frac{1}{4}(3-\mu^2)(\zeta-\alpha^2)Y=0 \tag{4.6.9}$$

and the boundary conditions become

$$Y(0) = \text{FINITE}$$

$$Y(1) = 0 \tag{4.6.10}$$

The solution of Eq. (4.6.9), a form of Heun's equation, which is finite at the origin is

$$Y(\zeta)=\sum_{n=0}^{\infty} C_n \zeta^n \tag{4.6.11}$$

with the C_n defined by

$$\beta_0 C_0 + \gamma_1 C_1 = 0$$

$$\alpha_{n-1}C_{n-1} + \beta_n C_n + \gamma_{n+1}C_{n+1} = 0 \; ; \; n \geq 1 \tag{4.6.12}$$

where

$$\alpha_n = n(n+2) + \tfrac{1}{4}(3-\mu^2)$$

$$\beta_n = -\alpha^2\left\{(2-\delta^2)n(n-1) + (4-2\delta^2)n + \tfrac{1}{4}(3-\mu^2)\right\}$$

$$\gamma_n = \alpha^4(1-\delta^2)n^2$$

(4.6.13)

In this case the eigenvalue problem is that of determining μ^2 such that

$$H(\mu^2) = \sum_{n=0}^{\infty} C_n(\mu^2) = 0$$

(4.6.14)

Numerical results for the eigenvalues for these special cases were obtained using methods similar to those described in Section (3.6). These results were compared with results obtained from the general program in the special cases $s=0$ and $s=1$ and complete agreement was found.

4.7 Results

The three Fortran computer programs referred to in this chapter were used to compute the first four eigenvalues, $n=0$ to 3, of the frequency parameter, μ^2, for values of the annulus radius ratio, δ, from 0.1 to 0.8, for values of the hub to disk radius ratio, ϵ, from 0.2 to 0.9, and for values of s from 0, to 3. The resulting data from these calculations were plotted and are presented in Figures 8 through 19. It should be noted that no calculations were carried out for cases in which ϵ was smaller than or equal to δ . This was considered a physically unrealizable case since it means that the center hole in the membrane is larger than or equal to the hub radius. For this reason ϵ was arbitrarily picked to be at least 10 percent greater than δ . The limits on ϵ and δ were governed somewhat by considerations of practicality in the computer programs. For values of δ less than 0.3 the convergence of the power series was extremely slow and required very long computer runs. For values of ϵ greater than 0.9, the fraction $\frac{1}{1-\epsilon^2}$ becomes large and leads to possible computational difficulties.

The plotted data, Figures 8 through 19, show that if the parameters δ , ϵ , and s are held constant the natural frequency of vibration increases as n increases. This behavior is predictable from the general theory of Sturm-Liouville eigenvalue problems.

The plots also show that the frequency of vibration is increased as the ratio of hub radius to disk radius, ϵ, is increased and δ , s, and n are held constant. The increase in the size of the hub with constant nodal geometry and disk size causes an overall stiffening of the disk, raising the natural frequencies of vibration.

In general, the frequency decreases as the annulus radius ratio, δ, is increased and s, n, and ϵ are constant. The exception for this rule is found in the case where the number of nodal diameters is much greater than the number of nodal circles. For this case, the frequency increases when δ is increased. This observation was also made and explained by Eversman [12] for the case of the annular elastic membrane with free edges.

The phenomenon is shown to be due to the change in the local stiffness in the membrane with changes in the nodal geometry. The stresses were known to be due only to the centripetal accelerations induced by the rotation. The stresses are given by Eqs. (4.2.5) and (4.2.6). These equations are repeated here for convenience:

$$\sigma_r = m\omega^2 \frac{(3+\nu)}{8} \frac{b^2}{\rho^2} (\rho^2 - 1)(\delta^2 - \rho^2)$$

$$\sigma_\theta = m\omega^2 \frac{(3+\nu)}{8} \frac{b^2}{\rho^2} \left[(1+\delta^2)\rho^2 + \delta^2 - \frac{1+3\nu}{3+\nu}\rho^2 \right]$$

We note that

$$\frac{d\sigma_r}{d\delta^2} = m\omega^2 \frac{(3+\nu)}{8} \frac{b^2}{\rho^2}(\rho^2-1) \quad < 0$$

$$\frac{d\sigma_\theta}{d\delta^2} = m\omega^2 \frac{(3+\nu)}{8} \frac{b^2}{\rho^2}(\rho^2+1) \quad > 0$$

so that for $0 \leq \rho \leq 1$ the radial stress decreases with increasing δ while the tangential stress increases with increasing δ .

For the case when the number of nodal diameters is substantially greater than the number of nodal circles the tangential stresses are predominantly more important in providing stiffness and the local stiffness in the membrane will increase with increasing δ . The increased stiffness causes increased frequency of vibration. If the number of nodal circles is substantially greater than the number of nodal diameters the radial stresses are dominant and the natural frequency tends to decrease with increasing δ [12]. If Figures 11, 14, and 17 are studied briefly it will be noted that the frequency of vibration is, in fact, influenced less by increasing or decreasing the hub radius in the cases where the number of nodal diameters is greater than the number of nodal circles. This observation can be explained. The larger number of nodal diameters causes sufficient restraint on the deflection of the membrane in the proximity of the inner edge that the increase or decrease in the hub size has little effect on the stiffness of the membrane or on the frequency of vibration.

4.8 Intermediate Cases of Central Clamping

The cases of full clamping and frictionless clamping represent the extremes of physically significant hub configurations. The fully clamped case corresponds to a disk built into the hub or shaft. It can be viewed as being equivalent to a disk clamped between collars with sufficient clamping pressure to prevent any radial deformation of the disk in the region of the hub. On the other extreme is the case of frictionless clamping in which there is no clamping pressure (alternatively, no hub-disk interface friction) and hence no constraint on the radial disk deformation in the hub region. Between these two configurations are the cases of intermediate clamping in which there is sufficient clamping pressure and hub-disk interface friction to prevent radial displacements of disk elements over a portion of the hub region, but not over the entire region. These cases have been studied in detail in the axisymmetric case by Bulkeley and Savage [10] and the extension to asymmetric vibrations constitutes only minor modifications of the procedures previously established in this report.

Bulkeley and Savage show that the form of the stress distribution in the spinning disk depends explicitly on the clamping conditions. In particular, for a disk of thickness h clamped by a hub of radius c with clamping pressure P and Coulomb friction coefficient μ, the radius r=g which limits the region of zero radial displacement is given by

$$g = \frac{2\mu P}{mh\omega^2}$$

with

$$0 \leq g \leq c$$

For values of $r<g$ there is no radial displacement while for $g \leq r \leq c$ the clamping pressure is insufficient to prevent slippage and radial displacement.

If the radius of effective clamping is between the inner radius of the disk and the radius of the hub, $a \leq g \leq c$, the stress distribution is given by

$$\sigma_r = \frac{3+\nu}{8} \frac{m\omega^2}{r^2}(b^2-r^2)(r^2+\epsilon_1 c^2) \tag{4.8.1}$$

$$\sigma_\theta = \frac{3+\nu}{8} \frac{m\omega^2}{r^2}\left\{(b^2-\epsilon_1 c^2)r^2 - \epsilon_1 c^2 b^2 - \frac{1+3\nu}{3+\nu} r^4\right\} \tag{4.8.2}$$

where

$$\epsilon_1 = \frac{\frac{1+3\nu}{3+\nu}\left(\frac{g}{c}\right)^2\left\{\frac{3+\nu}{1-\nu}\left(\frac{b}{g}\right)^2 - 1 + 8\left[\frac{1}{6}\left(\frac{c}{g}\right)^2 - \frac{1}{2}\left(\frac{c}{g}\right) + \frac{1}{3}\right]\right\}}{1 + \frac{1+\nu}{1-\nu}\left(\frac{b}{g}\right)^2} \tag{4.8.3}$$

and

a = inner annulus radius

b = outer disk radius

c = hub radius

g = radius at which clamping pressure and disk-hub interface friction become insufficient to prevent radial displacement of disk elements

μ = Coulomb friction at disk-hub interface

P = clamping pressure

h = disk thickness

It is also possible that the effective clamping radius will be less than the inner disk radius. If this occurs there will be slippage over the entire hub-disk interface. This will occur if

$$g < a$$

and the stress distribution will be

$$\sigma_r = \frac{3+\nu}{8} \frac{m\omega^2}{r^2}(b^2-r^2)(r^2+\epsilon_2 c^2) \tag{4.8.4}$$

$$\sigma_\theta = \frac{3+\nu}{8} \; \frac{m\omega^2}{r^2}\left\{(b^2-\epsilon_z c^2)r^2 - \epsilon_z c b^2 - \frac{1+3\nu}{3+\nu} r^4\right\}$$

(4.8.5)

where

$$\epsilon_z = -\frac{\left(\frac{a}{c}\right)^2\left\{\left[\left(\frac{b}{a}\right)^2-1\right] + \frac{4}{3}\,\frac{1-\nu}{3+\nu}\left(\frac{q}{a}\right)\left(1-\frac{c}{a}\right)\left[\frac{c}{a}\left(1+\frac{c}{a}\right) + \frac{2(2+\nu)}{1-\nu}\right]\right\}}{\left[\left(\frac{b}{a}\right)^2 - 1\right]}$$

(4.8.6)

It is recalled that in the case of frictionless clamping the stress distribution is given by

$$\sigma_r = \frac{3+\nu}{8} \; \frac{m\omega^2}{r^2}(b^2-r^2)(r^2-\delta^2 b^2)$$

(4.8.7)

$$\sigma_\theta = \frac{3+\nu}{8} \; \frac{m\omega^2}{r^2}\left\{(b^2+\delta^2 b^2) r^2 + \delta^2 b^4 - \frac{1+3\nu}{3+\nu} r^4\right\}$$

(4.8.8)

where

$$\delta^2 = \left(a/b\right)^2$$

Hence, if the computer program for the loosely clamped case is modified to replace

$$\delta^2 = -\left(\frac{c}{b}\right)^2 \epsilon \tag{4.8.9}$$

and the input to compute ϵ is provided, the vibration problem for the intermediate case with the effective clamping radius less than the inner annulus radius can be analyzed by using ϵ_2 in Eq. (4.8.9) and the case when the clamping radius is between the inner annulus radius and the hub radius can be analyzed by using ϵ_1, in Eq. (4.8.9).

Note that in the case P=0 we find

$$\epsilon_2 = -\left(\frac{a}{c}\right)^2$$

Since we replace

$$\delta^2 = -\left(\frac{c}{b}\right)^2 \epsilon_2 = \left(\frac{a}{b}\right)^2$$

we find that σ_r and σ_θ are properly defined for the loosely clamped case.

In the case g=c we have

$$\epsilon_1 = \frac{1-\nu}{3+\nu}\left[\frac{(3+\nu)b^2 - (1+\nu)c^2}{(1-\nu)c^2 + (1+\nu)b^2}\right]$$

and we replace

$$\delta^2 = -\left(\frac{c}{b}\right)^2 \epsilon_1$$

$$= -\frac{1-\nu}{3+\nu}\left(\frac{c}{b}\right)^2\left[\frac{(3+\nu)b^2-(1+\nu)c^2}{(1-\nu)c^2+(1+\nu)b^2}\right]$$

By referring to Eqs. (3.2.9) and (3.2.10) and accounting for the pertinent definition of δ^2 for this case given by Eq. (3.3.5) [note that the hub radius in this equation is a instead of c] we see that σ_r and σ_θ will be properly defined for the completely clamped case.

Hence it is concluded that the complete range of hub conditions between, and including, the extreme cases can be analyzed by a simple modification of the basic analysis for the loosely clamped case.

CHAPTER 5

THE EQUILIBRIUM STRESS AND DISPLACEMENT DISTRIBUTION IN A SPINNING SHALLOW SPHERICAL SHELL

5.1 Introduction

In Chapter 2 it was shown that the equations of motion for the small transverse vibrations of a spinning shallow spherical shell about its equilibrium configuration require the knowledge of the equilibrium stresses and displacements. It is the purpose of this chapter to present methods for the calculation of the equilibrium conditions for the cases of a freely spinning shell and a shell with central clamping. The linear and nonlinear theories of Reissner [30, 13, 14] are employed for this purpose.

In addition to providing calculation schemes, an evaluation is made regarding the adequacy of the use of membrane shell theory for the determination of the stresses and displacements. This was prompted by the paper of Johnson [8] in which was treated the problem of transverse vibrations of spinning shallow spherical membrane shells. In his analysis it is found that as the curvature of the undeformed shell is reduced, that is as the shell approaches a flat plate, the stress and displacement distribution, and hence the vibration natural frequencies, do not approach those of the flat plate. It is shown in the present analysis that this anomaly is due to the membrane assumption and that an extremely shallow spinning shell theory (approaching a flat plate) must include bending effects if a continuous transition from shell results to plate or flat membrane results is to be obtained. Furthermore, the inclusion of bending effects permits the specification of physically significant boundary conditions at the hub and outer edge.

5.2 The Equilibrium Equations for the Linear Theory

Equations (2.4.4) and (2.4.5), derived from a variational principle, correspond to Reissner's linear results [30] in the axisymmetric case with surface loading due to the centrifugal force effects. By introducing a stress function such that

$$N_{rr} = t\sigma_r = \frac{1}{r}\frac{dF}{dr} + \Omega$$

$$N_{\theta\theta} = t\sigma_\theta = \frac{d^2F}{dr^2} + \Omega$$

where

$$\Omega = -\frac{1}{2}\rho\omega^2 r^2$$

and by utilizing the compatibility equation of Eq. (2.4.7) we can obtain two simultaneous ordinary differential equations in the deflection and stress function

$$\nabla^2\nabla^2 F - \left(\frac{tE}{R}\right)\nabla^2 w = 2(1-\nu)\,\omega^2\rho \tag{5.2.1}$$

$$D\nabla^2\nabla^2 w + \left(\frac{1}{R}\right)\nabla^2 F = \frac{2\rho\omega^2}{R} r^2 \tag{5.2.2}$$

where

$$\nabla^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}$$

and

ρ = mt = mass per unit surface area

The boundary conditions for the case of the freely spinning shell are that the stresses and displacements be finite at the point r=0 and that the conditions at the free edge, r=b, are

$$N_{rr} = 0$$

$$\frac{d}{dr}(\nabla^2 w) = 0$$

$$\frac{d^2 w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr} = 0 \tag{5.2.3}$$

If there is a fully clamped central hub of radius a we must add to the free edge boundary conditions of Eqs. (5.2.3) the hub conditions

$$w(a) = 0$$

$$u(a) = 0$$

$$w_r(a) = 0 \tag{5.2.4}$$

5.3 The Equilibrium Equations for the Nonlinear Theory

The nonlinear theory of Reissner allows for the possibility of finite rotations of shell elements. As noted in Eqs. (2.4.6) and (2.4.9), the equilibrium equations are

$$D\left\{\phi_{rr} + \frac{\phi_r}{r} - \frac{\phi}{r^2}\right\} - \frac{\phi\psi}{r} + \frac{\psi}{r} = 0 \tag{5.3.1}$$

$$\frac{1}{tE}\left\{\psi_{rr} + \frac{\psi_r}{r} - \frac{\psi}{r^2}\right\} - \frac{\phi}{r} + \frac{1}{2}\,\frac{\phi^2}{r} = -\frac{(3+\nu)\rho\omega^2 r}{tE} \tag{5.3.2}$$

where

$$N_{rr} = t\sigma_r = \frac{\psi}{r}$$

$$N_{\theta\theta} = t\sigma_\theta = \psi_r + \rho\omega^2 r^2$$

$$\frac{dw}{dr} = \phi$$

The same boundary conditions apply in this case as do in the linear case, except that the vertical edge reaction (Kirchoff condition) is automatically satisfied and specification of w is not required unless it is desired to obtain w by integration.

5.4 Stresses and Displacements in the Freely Spinning Shell--Linear Theory

Because of the lack of utility for physically significant vibration problems, the case of the freely spinning shell with no hub has been treated only by the linear method except for the boundary layer analysis. The major result of this analysis is the demonstration of the continuous transition of the shell solutions to the flat plate solutions when the curvature approaches zero.

By following the method of Reissner [13], Eqs. (5.2.1) and (5.2.2) can be reduced to a single differential equation

$$\nabla^2\nabla^2(w+\bar{\lambda}F) - \left(\frac{i}{\ell^2}\right)\nabla^2(w+\bar{\lambda}F) = \frac{2\rho\omega^2}{RD} + 2\bar{\lambda}(1-\nu)\omega^2\rho \tag{5.4.1}$$

where

$$\ell^2 = \frac{Rt}{\sqrt{12(1-\nu^2)}}$$

$$\bar{\lambda} = \frac{i}{\sqrt{tED}}$$

and

$$i = \sqrt{-1}$$

Reissner obtains the solution to the homogeneous equation corresponding to Eq. (5.4.1) in the form

$$w = C_1 \, ber\frac{r}{\ell} + C_2 \, bei\frac{r}{\ell} + C_3 \, ker\frac{r}{\ell} + C_4 \, kei\frac{r}{\ell} + C_5 + C_7 \ln\frac{r}{\ell} \tag{5.4.2}$$

$$F = \frac{Et^2}{\sqrt{12(1-\nu^2)}}\left\{C_1 \,\mathrm{bei}\,\frac{r}{\ell} - C_2\,\mathrm{ber}\,\frac{r}{\ell} + C_3\,\mathrm{kei}\,\frac{r}{\ell} - C_4\,\mathrm{ker}\,\frac{r}{\ell} + C_6 \ln\frac{r}{\ell} + C_8\right\} \tag{5.4.3}$$

Of the eight constants in Eqs. (5.4.2) and (5.4.3), Reissner shows that C_8 can be excluded since only derivatives of F are significant and that C_7 must be excluded to preclude the possibility of circumferential displacement.

A particular solution to Eq. (5.4.1) which has the proper limiting behavior as the radius of curvature becomes infinite can be obtained in the form

$$w + \bar{\lambda}F = \sum_{n=0}^{\infty} a_n r^n \tag{5.4.4}$$

By substituting Eq. (5.4.4) into Eq. (5.4.1), solving for the unknown coefficients, and equating real and imaginary parts, there results for the particular solutions

$$w_p = \frac{(3+\nu)\rho\omega^2}{32}\frac{r^6}{RD}\sum_{n=0}^{\infty}\frac{(-1)^n}{[(2n+3)!]^2}\left(\frac{r}{2\ell}\right)^{4n} \tag{5.4.5}$$

$$F_p = \frac{(1-\nu)\rho\omega^2 r^4}{32} + \frac{(3+\nu)\rho\omega^2 r^4}{8}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{[(2n+2)!]^2}\left(\frac{r}{2\ell}\right)^{4n} \tag{5.4.6}$$

The general solution to Eq. (5.4.1) is the sum of the homogeneous solution, given by Eqs. (5.4.2) and (5.4.3), and the particular solution:

$$\begin{aligned} w &= C_1 \operatorname{ber}\frac{r}{\ell} + C_2 \operatorname{bei}\frac{r}{\ell} + C_3 \operatorname{ker}\frac{r}{\ell} + C_4 \operatorname{kei}\frac{r}{\ell} + C_5 \\ &\quad + \frac{3+\nu}{32}\,\frac{\rho\omega^2}{RD}\,r^6 \sum_{n=0}^{\infty}\frac{(-1)^n}{[(2n+3)!]^2}\left(\frac{r}{2\ell}\right)^{4n} \end{aligned} \tag{5.4.7}$$

$$\begin{aligned} F &= \frac{Et^2}{\sqrt{12(1-\nu^2)}}\left\{C_1 \operatorname{bei}\frac{r}{\ell} - C_2 \operatorname{ber}\frac{r}{\ell} + C_3 \operatorname{kei}\frac{r}{\ell} - C_4 \operatorname{ker}\frac{r}{\ell}\right. \\ &\quad \left. + C_6 \ln\frac{r}{\ell}\right\} + \frac{(1-\nu)\rho\omega^2 r^4}{32} + \frac{(3+\nu)\rho\omega^2 r^4}{8}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{[(2n+2)!]^2}\left(\frac{r}{2\ell}\right)^{4n} \end{aligned} \tag{5.4.8}$$

The constants in these solutions are to be determined from the boundary conditions. We choose to exclude the possibility of a rigid body vertical displacement by setting $C_5 = -C_1$. The Kirchoff condition

$$\frac{d}{dr}\nabla^2 w = 0 \quad at \quad r=b$$

leads directly to the conclusion that $C_6=0$. The requirement of finite stresses and displacements at the origin is violated if C_3 and C_4 are not zero, since $ker\frac{r}{\ell}$ is singular at $r=0$. Hence Eqs. (5.4.7) and (5.4.8) simplify to

$$w = C_1\left[ber\left(\frac{r}{\ell}\right)-1\right] + C_2\, bei\left(\frac{r}{\ell}\right)$$

$$+ \frac{3+\nu}{32}\frac{\rho\omega^2}{RD}r^6\sum_{n=0}^{\infty}\frac{(-1)^n}{[(2n+3)!]^2}\left(\frac{r}{2\ell}\right)^{4n} \tag{5.4.9}$$

$$F = \frac{Et^2}{\sqrt{12(1-\nu^2)}}\left\{C_1\, bei\left(\frac{r}{\ell}\right) - C_2\, ber\left(\frac{r}{\ell}\right)\right\}$$

$$+ \frac{(1-\nu)\rho\omega^2}{32}r^4 + \frac{(3+\nu)\rho\omega^2}{8}r^4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{[(2n+2)!]^2}\left(\frac{r}{2\ell}\right)^{4n} \tag{5.4.10}$$

The two constants C_1 and C_2 can be determined from the remaining two boundary conditions

$$N_{rr}(b) = 0$$

$$-\frac{M_{rr}}{D} = \omega_{rr} + \frac{\nu}{r}\omega_{r} = 0 \quad at \quad r = b$$

These two conditions lead to two simultaneous equations in the unknowns C_1 and C_2. Before writing these two equations, note that at r=b we have

$$\frac{r}{\ell} = \frac{b}{\ell} = \frac{\sqrt[4]{12(1-\nu^2)}}{\sqrt{\frac{Rt}{b^2}}} = \left[\frac{b^4 Et}{DR^2}\right]^{1/4}$$

From Eq. (2.2.1) with z=0 at r=b we find that

$$\sqrt{R^2-b^2} = R-h$$

By solving for R in terms of h and be we obtain

$$R = \frac{b^2+h^2}{2h}$$

Within the scope of shallow shell theory this becomes

$$R \approx \frac{b^2}{2h}$$

Hence we can write

$$\frac{b}{\ell} \approx 2\sqrt[4]{3(1-\nu^2)}\sqrt{h/t} = \lambda$$

(5.4.11)

With this observation we write the two equations in C_1 and C_2

$$\frac{Et^2}{\ell^2\sqrt{12(1-\nu^2)}}\left[C_1 \frac{bei'\lambda}{\lambda} - C_2 \frac{ber'\lambda}{\lambda}\right] + f_2(\lambda) = 0$$

$$\frac{1}{\ell^2}\left\{C_1\left[bei\,\lambda + (1-\nu)\frac{ber'\lambda}{\lambda}\right] - C_2\left[ber\,\lambda - (1-\nu)\frac{bei'\lambda}{\lambda}\right]\right.$$

$$\left. - \left[f_1''(\lambda) + \frac{\nu}{r} f_1'(\lambda)\right]\right\} = 0$$

where

$$f_1(r) = \frac{3+\nu}{32}\frac{\rho\omega^2}{RD} r^6 \sum_{n=0}^{\infty} \frac{(-1)^n}{[(2n+3)!]^2}\left(\frac{r}{2\ell}\right)^{4n}$$

$$f_2(r) = \frac{(1-\nu)\rho\omega^2}{32} r^4 + \frac{(3+\nu)\rho\omega^2}{8} r^4 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{[(2n+2)!]^2}\left(\frac{r}{2\ell}\right)^{4n}$$

and differentiations are denoted in the following way:

$$\frac{d f(r)}{d r} = f'(r)$$

$$\frac{d}{du} \, ber \, u = ber' \, u \quad , \text{ etc.}$$

If we introduce the notation

$$a_{11}(\lambda) = bei \, \lambda + (1-\nu) \frac{ber'\lambda}{\lambda} \qquad a_{21}(\lambda) = \frac{bei'\lambda}{\lambda}$$

$$a_{12}(\lambda) = ber \, \lambda - (1-\nu) \frac{bei'\lambda}{\lambda} \qquad a_{22}(\lambda) = \frac{ber'\lambda}{\lambda}$$

$$\phi_1(\lambda) = - f_1''(\lambda) - \frac{\nu}{r} f_1'(\lambda)$$

$$\phi_2(\lambda) = f_2(\lambda)$$

we obtain the set of equations

$$a_{11} C_1 - a_{12} C_2 = - \ell^2 \phi_1(\lambda)$$

$$a_{21} C_1 - a_{22} C_2 = - \frac{\ell^2 \sqrt{12(1-\nu^2)}}{E t^2} \phi_2(\lambda)$$

which have the solution

$$C_1(\lambda) = \frac{\ell^2 \phi_1(\lambda) \, a_{22}(\lambda) - \frac{\ell^2 \sqrt{12(1-\nu^2)}}{E t^2} \phi_2(\lambda) \, a_{12}(\lambda)}{a_{21}(\lambda) \, a_{12}(\lambda) - a_{11}(\lambda) \, a_{22}(\lambda)}$$

$$C_2(\lambda) = \frac{\ell^2 \phi_1(\lambda)\, a_{21}(\lambda) - \dfrac{\ell^2\sqrt{12(1-\nu^2)}}{Et^2}\, \phi_2(\lambda)\, a_{11}(\lambda)}{a_{21}(\lambda)\, a_{12}(\lambda) - a_{11}(\lambda)\, a_{22}(\lambda)}$$

A slightly more compact form of the solution can be had if we note that

$$\ell^2 \phi_1(\lambda) = -\frac{3+\nu}{32}\,\frac{\rho\omega^2\ell^6}{RD}\,\lambda^4 \sum_{n=0}^{\infty} \frac{(-1)^n\left[(4n+6)(4n+5)+\nu(4n+6)\right]}{\left[(2n+3)!\right]^2}\left(\frac{\lambda}{2}\right)^{4n}$$

$$\frac{\ell^2}{RD}\,\phi_2(\lambda) = -\frac{3+\nu}{8}\,\frac{\rho\omega^2\ell^6}{RD}\,\lambda^2 + \frac{3+\nu}{2}\,\frac{\rho\omega^2\ell^6}{RD}\,\lambda^2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n+1)}{\left[(2n+2)!\right]^2}\left(\frac{\lambda}{2}\right)^{4n}$$

We write these as

$$\ell^2 \phi_1(\lambda) = \frac{\rho\omega^2\ell^6}{RD}\,\lambda^2\,\bar{\phi}_1(\lambda)$$

$$\frac{\ell^2}{RD}\,\phi_2(\lambda) = \frac{\rho\omega^2\ell^6}{RD}\,\lambda^2\,\bar{\phi}_2(\lambda)$$

where

$$\bar{\phi}_1(\lambda) = -\frac{1}{32}\,\lambda^2 \sum_{n=0}^{\infty} \frac{(-1)^n\left[(4n+6)(4n+5)+\nu(4n+6)\right]}{\left[(2n+3)!\right]^2}\left(\frac{\lambda}{2}\right)^{4n}$$

$$\bar{\Phi}_2(\lambda) = -\frac{1}{8} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n+1)}{[(2n+2)!]^2} \left(\frac{\lambda}{2}\right)^{4n}$$

With these definitions the constants can be written

$$C_1(\lambda) = \frac{(3+\nu)\rho\omega^2 b^2}{RD} \ell^4 \psi_1(\lambda) \tag{5.4.12}$$

$$C_2(\lambda) = \frac{(3+\nu)\rho\omega^2 b^2}{RD} \ell^4 \psi_2(\lambda) \tag{5.4.13}$$

where

$$\psi_1(\lambda) = \frac{\bar{\Phi}_1(\lambda)\, a_{22}(\lambda) - \bar{\Phi}_2(\lambda)\, a_{12}(\lambda)}{a_{21}(\lambda) a_{12}(\lambda) - a_{11}(\lambda)\, a_{22}(\lambda)} \tag{5.4.14}$$

$$\psi_2(\lambda) = \frac{\bar{\Phi}_1(\lambda)\, a_{21}(\lambda) - \bar{\Phi}_2(\lambda)\, a_{11}(\lambda)}{a_{21}(\lambda)\, a_{12}(\lambda) - a_{11}(\lambda)\, a_{22}(\lambda)} \tag{5.4.15}$$

Equations (5.4.9) and (5.4.10) together with the constants defined by Eqs. (5.4.12), (5.4.13), (5.4.14), and (5.4.15), completely specify the stress and displacement distribution in the freely spinning shell. The stresses can be written explicitly by noting that since

$$N_{rr} = \frac{1}{r}\frac{dF}{dr} - \frac{1}{2}\rho\omega^2 r^2$$

$$N_{\theta\theta} = \frac{d^2F}{dr^2} - \frac{1}{2}\rho\omega^2 r^2$$

we can write, using Eqs. (5.4.9), (5.4.12), and (5.4.13),

$$N_{rr} = (3+\nu)\rho\omega^2 b^2 \Big\{ \psi_1(\lambda)\frac{\mathrm{bei}'\lambda(r/b)}{\lambda(r/b)} - \psi_2(\lambda)\frac{\mathrm{ber}'\lambda(r/b)}{\lambda(r/b)} - \frac{(3+\nu)}{8}\left(\frac{r}{b}\right)^2 + \frac{1}{2}\left(\frac{r}{b}\right)^2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n+1)}{[(2n+2)!]^2}\left(\frac{\lambda}{2}\frac{r}{b}\right)^{4n} \Big\} \tag{5.4.16}$$

$$N_{\theta\theta} = (3+\nu)\rho\omega^2 b^2 \Big\{ \psi_1(\lambda)\Big[\mathrm{ber}\,\lambda(r/b) - \frac{\mathrm{bei}'\lambda(r/b)}{\lambda(r/b)}\Big] + \psi_2(\lambda)\Big[\mathrm{bei}\,\lambda(r/b) + \frac{\mathrm{ber}'\lambda(r/b)}{\lambda(r/b)}\Big] - \frac{1}{8}\frac{(1+3\nu)}{(3+\nu)}\left(\frac{r}{b}\right)^2 + \frac{1}{8}\left(\frac{r}{b}\right)^2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(4n+4)(4n+3)}{[(2n+2)!]^2}\left(\frac{\lambda}{2}\frac{r}{b}\right)^{4n} \Big\} \tag{5.4.17}$$

The normal displacement w can be obtained by substituting Eqs. (5.4.12) and (5.4.13) in Eq. (5.4.9):

$$w = \frac{(3+\nu)\rho\omega^2 R^2}{E\sqrt{12(1-\nu^2)}} \Big\{ \lambda^2 \Big[\psi_1(\lambda)\big(\mathrm{ber}\ \lambda(r/b) - 1\big) + \psi_2(\lambda)\ \mathrm{bei}\ \lambda(r/b)\Big]$$

$$+ \frac{1}{32}\left(\lambda\frac{r}{b}\right)^6 \sum_{n=0}^{\infty} \frac{(-1)^n}{[(2n+3)!]^2}\left(\frac{\lambda}{2}\frac{r}{b}\right)^{4n}\Big\} \quad (5.4.18)$$

Although not required for the dynamic problem, the calculation of the shell bending moments can be accomplished by noting that they are defined by

$$M_{rr} = -D\left(w_{rr} + \nu\frac{w_r}{r}\right)$$

$$M_{\theta\theta} = -D\left(\frac{w_r}{r} + \nu w_{rr}\right)$$

By performing the appropriate operations using Eq. (5.4.18) we obtain for the bending moments

$$M_{rr} = \frac{(3+\nu)\rho\omega^2 b^2 \ell^2}{R}\Big\{ \psi_1(\lambda)\Big[\mathrm{bei}\ \lambda(r/b) + (1-\nu)\frac{\mathrm{ber}'\lambda(r/b)}{\lambda(r/b)}\Big]$$

$$- \psi_2(\lambda)\Big[\mathrm{ber}\ \lambda(r/b) - (1-\nu)\frac{\mathrm{bei}'\lambda(r/b)}{\lambda(r/b)}\Big]$$

$$- \frac{1}{32}\lambda^2\left(\frac{r}{b}\right)^4 \sum_{n=0}^{\infty}\frac{(-1)^n[(4n+6)(4n+5)+\nu(4n+6)]}{[(2n+3)!]^2}\left(\frac{\lambda}{2}\frac{r}{b}\right)^{4n}\Big\} \quad (5.4.19)$$

$$M_{\theta\theta} = -\frac{(3+\nu)\rho\omega^2 b^2 \ell^2}{R}\Big\{\psi_1(\lambda)\Big[(1-\nu)\frac{\mathrm{ber}'\lambda(r/b)}{\lambda(r/b)} - \nu\,\mathrm{bei}\,\lambda(r/b)\Big]$$

$$+\psi_2(\lambda)\Big[(1-\nu)\frac{\mathrm{bei}'\lambda(r/b)}{\lambda(r/b)} + \nu\,\mathrm{ber}\,\lambda(r/b)\Big]$$

$$+\frac{1}{32}\lambda^4\left(\frac{r}{b}\right)^4\sum_{n=0}^{\infty}\frac{(-1)^n\left[(4n+6)+\nu(4n+6)(4n+5)\right]}{\left[(2n+3)!\right]^2}\left(\frac{\lambda}{2}\frac{r}{b}\right)^{4n}\Big\} \tag{5.4.20}$$

5.5 The Limiting Case of the Freely Spinning Flat Disk

Consider the case when the radius of curvature becomes very large, or, alternatively when the parameter λ, defined by

$$\lambda = 2\sqrt[4]{3(1-\nu^2)}\sqrt{h/t}$$

where h/t is the ratio of the shell rise to the shell thickness, is very small. It can be verified that for small λ

$$a_{11}(\lambda) = \mathrm{bei}\,\lambda + (1-\nu)\frac{\mathrm{ber}'\lambda}{\lambda} \approx \frac{(3+\nu)}{16}\lambda^2$$

$$a_{12}(\lambda) = \mathrm{ber}\,\lambda - (1-\nu)\frac{\mathrm{bei}'\lambda}{\lambda} \approx \frac{1+\nu}{2}$$

$$a_{21}(\lambda) = \frac{\mathrm{bei}'\lambda}{\lambda} \approx \frac{1}{2}$$

$$a_{22}(\lambda) = \frac{\mathrm{ber}'\lambda}{\lambda} \approx -\frac{\lambda^2}{16}$$

$$a_{21}(\lambda)\,a_{12}(\lambda) - a_{11}(\lambda)\,a_{22}(\lambda) \approx \frac{1+\nu}{4}$$

$$\bar{\Phi}_1(\lambda) = -\frac{\lambda^2}{32}\sum_{n=0}^{\infty}\frac{(-1)^n\left[(4n+6)(4n+5)+\nu(4n+6)\right]}{\left[(2n+3)!\right]^2}\left(\frac{\lambda}{2}\right)^{4n}$$

$$\approx -\frac{5+\nu}{192}\lambda^2$$

$$\bar{\Phi}_2(\lambda) \approx -\frac{1}{8}$$

$$\bar{\Phi}_1(\lambda)\,a_{22}(\lambda) - \bar{\Phi}_2(\lambda)\,a_{12}(\lambda) \approx \frac{1+\nu}{16}$$

$$\bar{\Phi}_1(\lambda)\,a_{21}(\lambda) - \bar{\Phi}_2(\lambda)\,a_{11}(\lambda) \approx \frac{4+2\nu}{384}\lambda^2$$

It is seen then that

$$\lim_{\lambda\to 0}\psi_1(\lambda) = \frac{1}{4}$$

$$\lim_{\lambda\to 0}\psi_2(\lambda) = 0$$

Furthermore, for small λ

$$\frac{\mathrm{bei}'\lambda(r/b)}{\lambda(r/b)} \approx \frac{1}{2}$$

$$\frac{ber' \lambda(r/b)}{\lambda(r/b)} \approx -\frac{1}{16}\left(\lambda \frac{r}{b}\right)^2$$

$$bei\ \lambda(r/b) \approx \frac{1}{4}\left(\lambda \frac{r}{b}\right)^2$$

$$ber\ \lambda(r/b) \approx 1$$

By referring to Eqs. (5.4.16) and (5.4.17), it can thus be seen that

$$\lim_{\lambda \to 0} N_{rr} = \frac{3+\nu}{8} \rho \omega^2 b^2 \left[1 - \left(\frac{r}{b}\right)^2\right] \tag{5.5.1}$$

$$\lim_{\lambda \to 0} N_{\theta\theta} = \frac{3+\nu}{8} \rho \omega^2 b^2 \left[1 - \frac{1+3\nu}{3+\nu}\left(\frac{r}{b}\right)^2\right] \tag{5.5.2}$$

Equations (5.4.21) and (5.4.22) agree with the results of Timoshenko and Goodier [28] for the case of the freely spinning disk. Furthermore, by referring to Eqs. (5.4.18), (5.4.19), and (5.4.20) it can easily be verified that

$$\lim_{\substack{\lambda \to 0 \\ R \to \infty}} w = 0 \tag{5.5.3}$$

$$\lim_{\substack{\lambda \to 0 \\ R \to \infty}} M_{RR} = 0 \tag{5.5.4}$$

$$\lim_{\substack{\lambda \to 0 \\ R \to \infty}} M_{\theta\theta} = 0 \tag{5.5.5}$$

Hence, it is seen that the bending theory provides displacement and stress distributions which show a continuous transition to the flat disk results as $\lambda \to 0$ and $R \to \infty$.

5.6 Computational Results for the Freely Spinning Shell

The solutions for the direct stresses and normal displacement have been computed by direct numerical evaluation of Eqs. (5.4.16), (5.4.17) and (5.4.18). In addition, the outer surface bending stresses, defined by

$$t\sigma_{r_B} = B_{rr} = \frac{6M_{rr}}{t} \tag{5.6.1}$$

$$t\sigma_{\theta_B} = B_{\theta\theta} = \frac{6M_{\theta\theta}}{t} \tag{5.6.2}$$

have been calculated by direct evaluation of Eqs. (5.4.19) and (5.4.20). It should be noted that Eqs. (5.6.1) and (5.6.2) are somewhat unconventional in that they have the dimensions of force per unit length as do the present conventional definitions of the direct stresses. This definition was adopted to allow a direct comparison of the bending stresses with the direct stresses.

To furnish a basis for comparison, the corresponding results have been derived by using linear membrane theory. In the membrane case Eqs. (2.4.4) and (2.4.5) become

$$\frac{d}{dr}(r\sigma_r) - \sigma_\theta + m\omega^2 r^2 = 0 \tag{5.6.3}$$

$$\sigma_r + \sigma_\theta = m\omega^2 r^2 \tag{5.6.4}$$

The membrane solutions can readily be verified to be

$$N_{rr} = 0 \tag{5.6.5}$$

$$N_{\theta\theta} = \rho\,\omega^2 b^2 \left(\frac{r}{b}\right)^2 \tag{5.6.6}$$

$$\begin{aligned} w &= \frac{3+\nu}{2}\,\frac{R\rho\,\omega^2 b^2}{tE}\left(\frac{r}{b}\right)^2 \\ &= \frac{(3+\nu)\rho\,\omega^2 R^2}{E\sqrt{12(1-\nu^2)}}\left[\frac{1}{2}\left(\lambda\frac{r}{b}\right)^2\right] \end{aligned} \tag{5.6.7}$$

$$B_{rr} = -\frac{6}{\sqrt{12(1-\nu^2)}}\,\frac{(1+\nu)}{\lambda^2}\,(3+\nu)\rho\,\omega^2 b^2 \tag{5.6.8}$$

$$B_{\theta\theta} = -\frac{6}{\sqrt{12(1-\nu^2)}}\,\frac{(1+\nu)}{\lambda^2}\,(3+\nu)\rho\,\omega^2 b^2 \tag{5.6.9}$$

The results of the bending theory computations are shown in Figures 20 through 24. We have used several values of λ varying from the nearly flat configuration to the shell with a fairly substantial curvature (or alternatively, very thin). The results shown were obtained with $\nu = 0.20$. However, Poisson's ratio does not contribute strongly to the character of the solutions. These results can readily be compared to the membrane theory of Eqs. (5.6.5) through (5.6.9).

The transition with decreasing λ from a solution which is very close to the membrane results to one which practically coincides with the flat plate theory is clearly evident. It is noted that for high λ the bending stresses are small and show little variation except near the edge. By referring to Eqs. (5.6.8) and (5.6.9) it can be seen that membrane theory accurately predicts the magnitude of the bending stress in the interior of the disk but, of course, fails to do so at the edge since it is not accounted for in the boundary conditions. With decreasing λ the bending stresses first increase, then decrease as the thickness, or bending, effect becomes predominant and then the decrease in initial curvature dominates.

The direct stresses show a similar trend with decreasing λ. For high λ the direct radial stress is nearly zero while the tangential stress displays the parabolic character predicted by the membrane theory of Eq. (5.6.6). With decreasing λ we see a monotonic transition of the membrane type solution to the flat plate solution. It is particularly interesting to note the radical redistribution of the direct tangential stress as λ is decreased.

A similar observation regarding the deflection ω is also made. The characteristic parabolic solution of the membrane solution is seen for high λ, while zero normal deflection is approached for small λ, in which case the inertia load tends only to stretch the midsurface.

From these computations the importance of the bending contribution is seen and it is concluded that Johnson's results [8] may lead to serious errors. It is concluded that for values of λ which are not large the full bending theory should be employed for computation of the equilibrium state even if a membrane theory is subsequently used for the vibration analysis.

5.7 The Limiting Case of Large λ -Existence of Boundary Layer Phenomenon at Outer Edge

It is noted in Figures 20 to 24 that as λ becomes large the direct stresses N_{rr} and $N_{\theta\theta}$ and the displacement approach those predicted from the membrane theory. The bending stresses B_{rr} and $B_{\theta\theta}$ which are calculated from membrane theory are seen to approach those computed by the general theory for large λ in the interior of the shell, but deviate significantly near the edge of the shell. In fact, we note that the membrane theory is incapable of satisfying the zero bending moment condition at the outer edge, and this accounts for the deviation between the two solutions in this region. Since for large λ the deviation between the general theory and the membrane theory is localized near the outer edge, a boundary layer phenomenon is suggested. Reissner [14, 38] has dealt extensively with problems of this type for laterally loaded shallow spherical shells. It is the purpose of this section to demonstrate the existence of this phenomenon for the case of the spinning shell.

Because of the ease with which a nonlinear membrane solution can be obtained, it is convenient to investigate this phenomenon by using Reissner's nonlinear equations for the spinning shell given by Eqs. (2.4.8) and (2.4.9) and repeated here

$$D\left\{\Phi_{rr} + \frac{\Phi_r}{r} - \frac{\Phi}{r^2}\right\} - \frac{\Phi\Psi}{r} + \frac{\Psi}{R} = 0 \tag{5.7.1}$$

$$\frac{1}{Et}\left\{\Psi_{rr} + \frac{\Psi_r}{r} - \frac{\Psi}{r^2}\right\} - \frac{\Phi}{R} + \frac{1}{2}\frac{\Phi^2}{r} = -\frac{(3+\nu)}{Et}\rho\omega^2 r \tag{5.7.2}$$

where

$$r N_{rr} = \Psi \tag{5.7.3}$$

$$N_{\theta\theta} = \frac{d\psi}{dr} + \rho\omega^2 r^2 \tag{5.7.4}$$

$$\frac{dw}{dr} = \phi \tag{5.7.5}$$

The boundary conditions are that the solution be regular and symmetric at the origin and that at the outer edge, r=b

$$-\frac{M_{rr}}{D} = \phi_r + \nu\frac{\phi}{r} = 0$$

$$N_{rr} = 0$$

It is convenient to non-dimensionalize equations (5.7.1) and (5.7.2) by introducing

$$x = \mu\, r/b$$
$$\phi = \phi_0\, g(x)$$
$$\psi = \psi_0\, f(x)$$

in which case we obtain

$$D\phi_o\left(\frac{\mu}{b}\right)^2\left[g_{xx}+\frac{g_x}{x}-\frac{g}{x^2}\right]-\phi_o\psi_o\left(\frac{\mu}{b}\right)\frac{fg}{x}+\frac{\psi_o}{R}f=0$$

$$\frac{\psi_o}{Et}\left(\frac{\mu}{b}\right)^2\left[f_{xx}+\frac{f_x}{x}-\frac{f}{x^2}\right]-\frac{\phi_o}{R}g+\frac{1}{2}\left(\frac{\mu}{b}\right)\phi_o^2\frac{g^2}{x}=-\frac{(3+\nu)}{Et}\rho\omega^2\left(\frac{b}{\mu}\right)x$$

If we let

$$\frac{\psi_o}{Et}\left(\frac{\mu}{b}\right)^2=\frac{\phi_o}{R}$$

$$\phi_o=\left(\frac{b}{\mu}\right)\left(\frac{1}{R}\right)\left(\frac{3+\nu}{Et}\rho\omega^2R^2\right)=\left(\frac{b}{\mu}\right)\left(\frac{1}{R}\right)\gamma$$

where

$$\gamma=\frac{3+\nu}{Et}\rho\omega^2R^2$$

then we have

$$\frac{DR^2}{Et}\left(\frac{\mu}{b}\right)^4\left[g_{xx}+\frac{g_x}{x}-\frac{g}{x^2}\right]+f-\gamma\frac{fg}{x}=0 \tag{5.7.6}$$

$$f_{xx} + \frac{f_x}{x} - \frac{f}{x^2} - g + \frac{1}{2}\gamma\frac{g^2}{x} = -x$$

(5.7.7)

It is seen that γ is a measure of the nonlinearity of the problem.

The stresses and displacements will become

$$N_{rr} = \frac{1}{\mu^2}\left(\frac{b}{R}\right)^2 E t \, \gamma \frac{f(x)}{x} = \frac{1}{\mu^2}(3+\nu)\rho\omega^2 b^2 \frac{f(x)}{x}$$

$$N_{\theta\theta} = \frac{\mu}{b}\psi_o f'(x) + m\omega^2\left(\frac{b}{\mu}\right)^2 x^2 = \frac{1}{\mu^2}\left(\frac{b}{R}\right)^2$$

$$= \frac{1}{\mu^2}(3+\nu)\rho\omega^2 b^2\left[f'(x) + \frac{x^2}{3+\nu}\right]$$

$$\frac{dw}{dx} = \left(\frac{b}{\mu}\right)\phi = \left(\frac{b}{\mu}\right)\phi_o g(x) = \left(\frac{b}{\mu}\right)^2\frac{\gamma}{R} g(x)$$

By expanding and using shallow shell relations and the definition of λ we obtain for dw/dx

$$\frac{dw}{dx} = \frac{1}{\mu^2} \frac{(3+\nu)\rho\omega^2 R^2}{E\sqrt{12(1-\nu^2)}} \lambda^2 g(x)$$

The bending moments will then become

$$-M_{rr} = \frac{D\gamma}{R}\left[g_x + \nu\frac{g}{x}\right] = \frac{(3+\nu)\rho\omega^2 b^2 \ell^2}{R}\left(\frac{1}{\lambda^2}\right)\left[g_x + \nu\frac{g}{x}\right]$$

$$-M_{\theta\theta} = \frac{D\gamma}{R}\left[\frac{g}{x} + \nu g_x\right] = \frac{(3+\nu)\rho\omega^2 b^2 \ell^2}{R}\left(\frac{1}{\lambda^2}\right)\left[\frac{g}{x} + \nu g_x\right]$$

The boundary conditions on the non-dimensional form of the problem will be that f(x) and g(x) be regular and symmetric at x=0 and that at x=1

$$g_x + \nu\frac{g}{x} = 0$$

$$f(x) = 0$$

Note in Eq. (5.7.6) that

$$\frac{1}{b^4}\frac{DR^2}{Et} \approx \frac{1}{48(1-\nu^2)}\frac{t}{h} = \frac{1}{\lambda^4}$$

Hence, we write the defining equations

$$\left(\frac{\mu}{\lambda}\right)^4\left[g_{xx}+\frac{1}{x}g_x-\frac{1}{x^2}g\right]+f-\frac{\gamma}{x}fg=0 \tag{5.7.8}$$

$$f_{xx}+\frac{1}{x}f_x-\frac{1}{x^2}f-g+\frac{\gamma}{2x}g^2=-x \tag{5.7.9}$$

To investigate the boundary layer phenomenon which exists for large λ we make use of some of the concepts of perturbation theory [39, 40]. Since the outer edge is the region of primary interest we will consider "inner" and "outer" expansions relative to that point. A suitable outer variable is x with $\mu=1$, in which case Eqs. (5.7.8) and (5.7.9) become

$$\epsilon^4\left[g_{xx}+\frac{1}{x}g_x-\frac{1}{x^2}g\right]+f-\frac{\gamma}{x}fg=0 \tag{5.7.10}$$

$$f_{xx}+\frac{1}{x}f_x-\frac{1}{x^2}f-g+\frac{\gamma}{2x}g^2=-x \tag{5.7.11}$$

where ϵ^4 is assumed to be a small parameter defined by

$$\epsilon^4 = \frac{1}{\lambda^4}$$

The first term in an asymptotic series in powers of ϵ will be defined by the equations

$$f - \frac{\gamma}{x} f g = 0 \tag{5.7.12}$$

$$f_{xx} + \frac{1}{x} f_x - \frac{1}{x^2} f - g + \frac{1}{2}\frac{\gamma}{x} g^2 = -x \tag{5.7.13}$$

A solution to this degenerate set of equations which satisfies the boundary conditions at x=0 is taken as

$$f_m(x) = 0 \tag{5.7.14}$$

$$g_m(x) = \frac{x}{g}\left[1 - \sqrt{1-2\gamma}\right] \tag{5.7.15}$$

This is the nonlinear membrane solution in the case when the membrane shell is not deflected to a flat plate. We will assume in this analysis that $2\gamma < 1$, the condition that the deformed membrane shell is not flat, so that the above form of outer solution is valid. It is noted that Eqs. (5.7.14) and (5.7.15) will not satisfy the boundary condition of zero moment at x=1, since it is found that

$$M_{rr} = -\frac{D}{R}(1+\nu)\,\delta$$

where

$$\delta = 1 - \sqrt{1-2\gamma}$$

To investigate the nature of the solution in the neighborhood of the free edge at x=1 it proves convenient to recast the problem by defining

$$g(x) = g_m(x) + G(x)$$

$$f(x) = F(x)$$

In this case Eqs. (5.7.8) and (5.7.9) become

$$\left(\frac{\mu}{\lambda}\right)^4\left[G_{xx} + \frac{1}{x}G_x - \frac{1}{x^2}G\right] + (1-\delta)F - \gamma\frac{FG}{x} = 0 \tag{5.7.16}$$

$$F_{xx} + \frac{1}{x}F_x - \frac{1}{x^2}F - (1-\delta)G + \frac{\gamma}{2x}G^2 = 0 \tag{5.7.17}$$

The new boundary conditions require that F and G be regular and symmetric at x=0, and that at the free edge

$$G_x + \nu \frac{G}{x} = -(1+\nu)\frac{\delta}{\gamma}$$

$$F = 0$$

The inner problem is formulated by choosing the parameter μ to be

$$\mu = \lambda = \frac{1}{\epsilon}$$

in which case

$$x = \frac{1}{\epsilon}\frac{r}{b}$$

The outer edge is made the origin by the change of variable

$$\xi = \frac{1}{\epsilon}(1-\epsilon x) = \frac{1}{\epsilon}(1 - r/b)$$

Equations (5.7.16) and (5.7.17) then become

$$G_{\xi\xi} - \frac{\epsilon}{1-\epsilon\xi}G_\xi - \frac{\epsilon^2}{(1-\epsilon\xi)^2}G + (1-\delta)F - \frac{\gamma\epsilon FG}{1-\epsilon\xi} = 0 \tag{5.7.18}$$

$$F_{\xi\xi} - \frac{\epsilon}{1-\epsilon\xi}F_\xi - \frac{\epsilon^2}{(1-\epsilon\xi)^2}F - (1-\delta)G + \frac{\gamma}{2}\frac{\epsilon G^2}{1-\epsilon\xi} = 0 \tag{5.7.19}$$

The boundary conditions are

$\xi = 0:$

$$G_\xi - \frac{\nu \epsilon G}{1-\epsilon\xi} = (1+\nu)\frac{\delta}{\gamma}$$

$$F = 0$$

$\xi = \frac{1}{\epsilon}:$ F and G regular and symmetric

If ξ is 0(1), that is if we restrict ourselves to a narrow region in which $1 - r/b$ is $0(\epsilon)$, then Eqs. (5.7.18) and (5.7.19) and the boundary conditions can be approximated to within terms of $O(\epsilon)$ by

$$\frac{d^2G}{d\xi^2} + (1-\delta)F = 0 \tag{5.7.20}$$

$$\frac{d^2F}{d\xi^2} - (1-\delta)G = 0 \tag{5.7.21}$$

$\xi = 0:$ $\quad \frac{dG}{d\xi} = (1+\nu)\frac{\delta}{\gamma}$

$$F = 0$$

$\xi \to \infty:$ F, G REGULAR AND SYMMETRIC

Equations (5.7.20) and (5.7.21) can be construed as defining the term of $O(\epsilon^{\circ})$ in an asymptotic series in powers of the small parameter ϵ . In principle, the higher order terms could be sequentially obtained and the procedure of matching with the outer solution could be accomplished. However, we have succeeded in demonstrating that there does in fact exist a narrow region in which the solution departs radically from the outer solution and can be made to satisfy the free edge condition.

From the above analysis and the behavior shown by the computed solutions, it is concluded that for large λ the solution is characteristic of membrane theory over the interior of the shell but in a narrow region at the edge it will involve significant bending to allow satisfaction of the free edge condition.

A complete analysis of the boundary layer phenomenon will be undertaken as an extension of the current effort.

5.8 Linear Theory for the Case of a Fully Clamped Central Hub

In the case of a fully clamped central hub the boundary conditions to be satisfied by the general solution given by Eqs. (5.4.7) and (5.4.8) are

$$r = a: \quad w = 0$$
$$u = 0$$
$$w_r = 0 \tag{5.8.1}$$

$$r = b: \quad N_{rr} = 0$$
$$\frac{d}{dr}\nabla^2 w = 0$$
$$w_{rr} + \frac{\nu}{r} w_r = 0 \tag{5.8.2}$$

The Kirchoff condition again requires that C_6 vanish. The general solution then becomes

$$w^* = C_1 \operatorname{ber}\frac{r}{\ell} + C_2 \operatorname{bei}\frac{r}{\ell} + C_3 \operatorname{ker}\frac{r}{\ell} + C_4 \operatorname{kei}\frac{r}{\ell} + C_5 + \frac{2(3+\nu)}{\lambda^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{[(2n+3)!]^2}\left(\frac{r}{2\ell}\right)^{4n+6} \tag{5.8.3}$$

$$F^* = \frac{1}{\lambda^2}\left[C_1 \operatorname{bei}\frac{r}{\ell} - C_2 \operatorname{ber}\frac{r}{\ell} + C_3 \operatorname{kei}\frac{r}{\ell} - C_4 \operatorname{ker}\frac{r}{\ell}\right] + \frac{1-\nu}{32}\left(\frac{r}{b}\right)^4 + \frac{2(3+\nu)}{\lambda^4}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{[(2n+2)!]^2}\left(\frac{r}{2\ell}\right)^{4n+4} \tag{5.8.4}$$

$$N_{rr}^* = \frac{1}{(r/\ell)}\left[C_1 \operatorname{bei}'\left(\frac{r}{\ell}\right) - C_2 \operatorname{ber}'\left(\frac{r}{\ell}\right) + C_3 \operatorname{kei}'\left(\frac{r}{\ell}\right) - C_4 \operatorname{ker}'\left(\frac{r}{\ell}\right)\right] + \frac{2(3+\nu)}{\lambda^2} g_1(r/\ell) \tag{5.8.5}$$

$$N_{\theta\theta}^* = C_1\left[\mathrm{ber}(r/\ell) - \frac{\mathrm{bei}'(r/\ell)}{r/\ell}\right] + C_2\left[\mathrm{bei}(r/\ell) + \frac{\mathrm{ber}'(r/\ell)}{r/\ell}\right]$$

$$+ C_3\left[\mathrm{ker}(r/\ell) - \frac{\mathrm{kei}'(r/\ell)}{r/\ell}\right] + C_4\left[\mathrm{kei}(r/\ell) + \frac{\mathrm{ker}'(r/\ell)}{r/\ell}\right]$$

$$+ \frac{2(3+\nu)}{\lambda^3} g_3(r/\ell) \qquad (5.8.6)$$

$$M_{rr}^* = \frac{1}{\sqrt{12(1-\nu^2)}}\left\{C_1\left[\mathrm{bei}(r/\ell) + (1-\nu)\frac{\mathrm{ber}'(r/\ell)}{r/\ell}\right]\right.$$

$$- C_2\left[\mathrm{ber}(r/\ell) - (1-\nu)\frac{\mathrm{bei}'(r/\ell)}{r/\ell}\right] + C_3\left[\mathrm{kei}(r/\ell) + (1-\nu)\frac{\mathrm{ker}'(r/\ell)}{r/\ell}\right]$$

$$\left. - C_4\left[\mathrm{ker}(r/\ell) - (1-\nu)\frac{\mathrm{kei}'(r/\ell)}{r/\ell}\right] - \frac{3+\nu}{\lambda^2} g_2(r/\ell)\right\} \qquad (5.8.7)$$

$$M_{\theta\theta}^* = \frac{1}{\sqrt{12(1-\nu^2)}}\left\{C_1\left[\nu\,\mathrm{bei}(r/\ell) - (1-\nu)\frac{\mathrm{ber}'(r/\ell)}{r/\ell}\right]\right.$$

$$- C_2\left[\nu\,\mathrm{ber}(r/\ell) + (1-\nu)\frac{\mathrm{bei}'(r/\ell)}{r/\ell}\right] + C_3\left[\nu\,\mathrm{kei}(r/\ell) - (1-\nu)\frac{\mathrm{ker}'(r/\ell)}{r/\ell}\right]$$

$$\left. - C_4\left[\nu\,\mathrm{ker}(r/\ell) + (1-\nu)\frac{\mathrm{kei}'(r/\ell)}{r/\ell}\right] - \frac{3+\nu}{\lambda^2} g_4(r/\ell)\right\} \qquad (5.8.8)$$

$$u^* = \frac{\rho \omega^2 b^2}{Et}\left(\frac{r}{b}\right)\Big\{ -C_1\left[(1+\nu)\frac{bei'(r/\ell)}{(r/\ell)}\right] + C_2\left[(1+\nu)\frac{ber'(r/\ell)}{(r/\ell)}\right]$$

$$-C_3\left[(1+\nu)\frac{kei'(r/\ell)}{(r/\ell)}\right] + C_4\left[(1+\nu)\frac{ker'(r/\ell)}{(r/\ell)}\right] - C_5$$

$$+\left(\frac{r}{b}\right)^2 + \frac{2(3+\nu)}{\lambda^2}\left[g_3(r/\ell) - \nu g_1(r/\ell) - g_5(r/\ell)\right]\Big\} \qquad (5.8.9)$$

where

$$w = \frac{\rho \omega^2 b^2 R}{Et} w^*$$

$$F = \rho \omega^2 b^4 F^*$$

$$N_{rr} = \rho \omega^2 b^2 \; N_{rr}^*$$

$$N_{\theta\theta} = \rho \omega^2 b^2 \; N_{\theta\theta}^*$$

$$M_{rr} = \rho \omega^2 b^2 t \; M_{rr}^*$$

$$M_{\theta\theta} = \rho \omega^2 b^2 t \; M_{\theta\theta}^*$$

$$u = b u^*$$

$$g_1(r/\ell) = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}(n+1)}{[(2n+2)!]^2}\left(\frac{r}{2\ell}\right)^{4n+2}$$

$$g_2(r/\ell) = \sum_{n=0}^{\infty} \frac{(-1)^n(2n+3)(4n+5+\nu)}{[(2n+3)!]^2}\left(\frac{r}{2\ell}\right)^{4n+4}$$

$$g_3(r/\ell) = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}(n+1)(4n+3)}{[(2n+2)!]^2}\left(\frac{r}{2\ell}\right)^{4n+2}$$

$$g_4(r/\ell) = \sum_{n=0}^{\infty} \frac{(-1)^{n}(2n+3)[(4n+5)\nu+1]}{[(2n+3!]^2}\left(\frac{r}{2\ell}\right)^{4n+4}$$

$$g_5(r/\ell) = \sum_{n=0}^{\infty} \frac{(-1)^{n}}{[(2n+3)!]^2}\left(\frac{r}{2\ell}\right)^{4n+6}$$

$$g_6(r/\ell) = \sum_{n=0}^{\infty} \frac{(-1)^{n}(2n+3)}{[(2n+3)!]^2}\left(\frac{r}{2\ell}\right)^{4n+6}$$

The constants C_1 through C_5 are determined from the boundary conditions which lead to the following five algebraic equations (here written in the same order as the boundary conditions of Eqs. (5.8.1) and (5.8.2)):

$$C_5 = -C_1 \operatorname{ber}\lambda(a/b) - C_2 \operatorname{bei}\lambda(a/b) - C_3 \operatorname{ker}\lambda(a/b) - C_4 \operatorname{kei}\lambda(a/b) - \frac{2(3+\nu)}{\lambda^2} g_5\left(\frac{\lambda}{2}\frac{a}{b}\right) \qquad (5.8.10)$$

$$C_1\left[\operatorname{ber}\lambda(a/b) - (1+\nu)\frac{\operatorname{bei}'\lambda(a/b)}{\lambda(a/b)}\right] + C_2\left[(1+\nu)\frac{\operatorname{ber}'\lambda(a/b)}{\lambda(a/b)} + \operatorname{bei}\lambda(a/b)\right]$$

$$+ C_3\left[\operatorname{ker}\lambda(a/b) - \frac{(1+\nu)\operatorname{kei}'\lambda(a/b)}{\lambda(a/b)}\right] + C_4\left[(1+\nu)\frac{\operatorname{ker}'\lambda(a/b)}{\lambda(a/b)} + \operatorname{kei}\lambda(a/b)\right] =$$

$$-\left(\frac{a}{b}\right)^2 - \frac{2(3+\nu)}{\lambda^2}\left[g_3\left(\frac{\lambda}{2}\frac{a}{b}\right) - \nu g_1\left(\frac{\lambda}{2}\frac{a}{b}\right) - g_5\left(\frac{\lambda}{2}\frac{a}{b}\right)\right] \qquad (5.8.11)$$

$$C_1 \operatorname{ber}'\lambda(a/b) + C_2 \operatorname{bei}'\lambda(a/b) + C_3 \operatorname{ker}'\lambda(a/b) + C_4 \operatorname{kei}'\lambda(a/b) = - \frac{2(3+\nu)}{\lambda^2} g_6\left(\frac{\lambda}{2}\frac{a}{b}\right) \tag{5.8.12}$$

$$C_1 \operatorname{bei}'\lambda - C_2 \operatorname{ber}'\lambda + C_3 \operatorname{kei}'\lambda - C_4 \operatorname{ker}'\lambda = - \frac{2(3+\nu)}{\lambda} g_1\left(\frac{\lambda}{2}\right) \tag{5.8.13}$$

$$C_1\left[\operatorname{bei}\lambda + (1-\nu)\frac{\operatorname{ber}'\lambda}{\lambda}\right] - C_2\left[\operatorname{ber}\lambda - (1-\nu)\frac{\operatorname{bei}'\lambda}{\lambda}\right] + C_3\left[\operatorname{kei}\lambda + (1-\nu)\frac{\operatorname{ker}'\lambda}{\lambda}\right] - C_4\left[\operatorname{ker}\lambda - (1-\nu)\frac{\operatorname{kei}'\lambda}{\lambda}\right] = \frac{3+\nu}{\lambda^2} g_2\left(\frac{\lambda}{2}\right) \tag{5.8.14}$$

It is seen that C_1 through C_4 are defined by the four Eqs. (5.8.11) through (5.8.14) while C_5 is obtained from Eq. (5.8.10). As soon as the constants are determined, the direct stresses, bending moments, and normal deflection can be computed from Eqs. (5.8.5), (5.8.6), (5.8.7), (5.8.8), and (5.8.3), respectively. The outer surface bending stresses, defined by Eqs. (5.6.1) and (5.6.2) are directly obtainable from the bending stress results.

For demonstration purposes we have evaluated the stresses and displacements for the case of a hub with $(a/b) = 0.125$ and $\nu = 0.20$. The results, plotted for several values of λ, are given in Figures 25 through 40. The characteristics of the solution will be discussed in Section (5.10) where they are compared with the nonlinear results.

5.9 Nonlinear Theory for the Case of a Fully Clamped Central Hub

As will be seen in detail in Chapter 6, the nature of the spinning shell vibration problem is critically dependent on the stress and displacement distributions obtained from the equilibrium solutions. To assure that an accurate evaluation of the equilibrium state is available for the dynamic problem, extensive calculations have been performed, for the case of the clamped central hub, using Reissner's nonlinear theory [14] which allows finite rotations of shell elements. The linear and nonlinear results have been compared and the regions of parameter values in which the nonlinear effects are important have been established.

The governing equations of Reissner's theory were derived from a variational principle in Chapter 2, were used in the boundary layer analysis in Section (5.7), and are repeated here for reference:

$$D\left\{ \phi_{rr} + \frac{\phi_r}{r} - \frac{\phi}{r^2} \right\} - \frac{\phi\psi}{r} + \frac{\psi}{R} = 0 \tag{5.9.1}$$

$$\frac{1}{Et}\left\{ \psi_{rr} + \frac{\psi_r}{r} - \frac{\psi}{r^2} \right\} - \frac{\phi}{R} + \frac{1}{2}\frac{\phi^2}{R} = -\frac{(3+\nu)}{Et}\rho\omega^2 r \tag{5.9.2}$$

where

$$r N_{rr} = \psi \tag{5.9.3}$$

$$N_{\theta\theta} = \frac{d\psi}{dr} + \rho\omega^2 r^2 \tag{5.9.4}$$

$$\frac{dw}{dr} = \phi \tag{5.9.5}$$

As shown in Section 5.7 , a non-dimensional form of these equations can be written

$$\frac{1}{\lambda^4}\left[g_{xx} + \frac{g_x}{x} - \frac{g}{x^2} \right] + f - \gamma\frac{fg}{x} = 0 \tag{5.9.6}$$

$$f_{xx} + \frac{f_x}{x} - \frac{f}{x^2} - g + \frac{1}{2}\gamma\frac{g^2}{x} = -x \tag{5.9.7}$$

where

$$x = r/b$$

$$\phi = \phi_o\, g(x)$$

$$\psi = \psi_o\, f(x)$$

and

$$\phi_o = \left(\frac{b}{R}\right)\gamma$$

$$\psi_o = \frac{Etb^3}{R^2}\gamma$$

$$\gamma = \frac{(3+\nu)}{Et}\rho\omega^2 R^2$$

$$\frac{1}{\lambda^4} = \frac{DR^2}{b^4 Et}$$

The functions f and g are defined such that

$$N_{rr} = (3+\nu)\rho\omega^2 b^2 \frac{f(x)}{x} \tag{5.9.8}$$

$$N_{\theta\theta} = (3+\nu)\rho\omega^2 b^2\left[f'(x) + \frac{x^2}{3+\nu}\right] \tag{5.9.9}$$

$$\frac{dw}{dx} = \frac{(3+\nu)}{Et}\frac{b^2}{R}\rho\omega^2 R^2 g(x) \tag{5.9.10}$$

Equations (5.9.8), (5.9.9), and (5.9.10) can be written in the slightly modified form

$$\sigma_r = \frac{N_{rr}}{t} = \frac{(3+\nu)\rho\omega^2 R^2}{Et} E\left(\frac{b}{R}\right)^2 \frac{f(x)}{x}$$

$$= \gamma E\left(\frac{b}{R}\right)^2 \frac{f(x)}{x} \tag{5.9.11}$$

$$\sigma_\theta = \frac{N_{\theta\theta}}{t} = \frac{(3+\nu)\rho\omega^2 R^2}{Et} E\left(\frac{b}{R}\right)^2 \left[f'(x) + \frac{x^2}{3+\nu}\right]$$

$$= \gamma E\left(\frac{b}{R}\right)^2 h(x) \tag{5.9.12}$$

$$\frac{R}{b^2}\frac{dw}{dx} = \frac{(3+\nu)\rho\omega^2 R^2}{Et} g(x)$$

$$= \gamma g(x) \tag{5.9.13}$$

where for convenience we have defined

$$h(x) = f'(x) + \frac{x^2}{3+\nu}$$

The definitions of Eqs. (5.9.11), (5.9.12), and (5.9.13) prove to be convenient when describing the equilibrium direct stress and displacement distributions for the vibration problem in Chapter 6.

The bending moments are defined by

$$M_{rr} = -\frac{(3+\nu)\rho\omega^2 b^2 l^2}{R}\left(\frac{1}{\lambda^2}\right)\left[g_x + \nu\frac{g}{x}\right] \tag{5.9.14}$$

$$M_{\theta\theta} = -\frac{(3+\nu)\rho\omega^2 b^2 l^2}{R}\left(\frac{1}{\lambda^2}\right)\left[g_x + \nu\frac{g}{x}\right] \tag{5.9.15}$$

The outermost surface bending stresses are given by [13]

$$\sigma_{r_B} = \pm 6\frac{M_{rr}}{t^2}$$

$$\sigma_{\theta_B} = \pm 6\frac{M_{\theta\theta}}{t^2}$$

By referring to Eqs. (5.9.14) and (5.9.15), and making use of the shallow shell approximations, it is found that

$$\sigma_{r_B} = \pm\frac{6(3+\nu)\rho\omega^2 b^2}{t}\frac{1}{\sqrt{12(1-\nu^2)}}\left(\frac{1}{\lambda^2}\right)\left[g_x + \nu\frac{g}{x}\right] \tag{5.9.16}$$

$$\sigma_{\theta_B} = \pm\frac{6(3+\nu)\rho\omega^2 b^2}{t}\frac{1}{\sqrt{12(1-\nu^2)}}\left(\frac{1}{\lambda^2}\right)\left[g_x + \nu\frac{g}{x}\right] \tag{5.9.17}$$

The boundary conditions accompanying Eqs. (5.9.6) and

(5.9.7) are

$$x = 1: \quad g_x + \frac{\nu}{x} g = 0 \qquad (M_{rr} = 0)$$

$$f = 0 \qquad (N_{rr} = 0) \tag{5.9.18}$$

$$x = a/b: \quad g = 0 \qquad \left(\frac{dw}{dr} = 0\right)$$

$$f_x + \frac{x^2}{3+\nu} - \frac{\nu}{x} f = 0 \qquad (u = 0) \tag{5.9.19}$$

Equations (5.9.6), (5.9.7), (5.9.18) and (5.9.19) constitute a nonlinear two-point boundary value problem. A method which is essentially a modification of Archer's technique [18] has been employed to obtain solutions for the stresses and axial component of displacement.

Archer's method is iterative in nature. He solves the linear portion of the problem and then utilizes this solution to evaluate the nonlinear terms. These are then treated as an additional nonhomogeneous contribution to the linear equations for the next solution step. This procedure is carried out for several iterations until successive solutions coincide to within a specified error. Archer's chief contribution is the finite difference scheme which he has found to be appropriate for solutions to the successive linear two point boundary value problems.

Archer's method, in its basic form, was initially employed to obtain solutions to the present problem. The grid employed 36 points in the interval from $X = a/b$ to $X = 1$ and represented nearly the limit in complexity which could be handled on the Wichita State University IBM 1620 digital computer. While some results obtained by this method were satisfactory, several instances occurred in which the iteration procedure diverged. Since a buckling phenomenon is not expected, it was concluded that the lack of convergence was due to some inadequacy in the technique. No attempt was made to identify the source of the convergence problem and it is possible that the particular implementation of the method was not adequate.

In view of the convergence problems encountered with Archer's basic technique, a slightly modified method was employed and convergence was obtained for all combinations of parameters used. This method, discussed more completely in Reference [41], employs the finite difference method of Archer, but defines the successive linear two point boundary value problems involved in the iteration in a different way. The method is best explained by considering a simple example. Assume that we have the simultaneous nonlinear equations

$$\phi'' + \psi = \alpha(x) + \phi^2$$

$$\psi'' + \phi = \phi\psi \tag{5.9.20}$$

The first step in the procedure would be to obtain ϕ_1, and ψ_1, solutions to the linear problem

$$\phi_1'' + \psi_1 = \alpha(x)$$

$$\psi_1'' + \phi_1 = 0 \tag{5.9.21}$$

We then write the true solution as the sum of ϕ_1, ψ_1, and some remainder terms ϕ_2 and ψ_2,

$$\phi = \phi_1 + \phi_2$$

$$\psi = \psi_1 + \psi_2 \tag{5.9.22}$$

By substituting Eqs. (5.9.22) into Eqs. (5.9.20) and making use of Eqs. (5.9.21) we obtain the governing equations for ϕ_2 and ψ_2:

$$\phi_2'' + \psi_2 - 2\phi_1\phi_2 = \phi_1^2 + \phi_2^2$$

$$\psi_2'' + \phi_2 - \phi_1\psi_2 - \psi_1\phi_2 = \phi_1\psi_1 + \phi_2\psi_2 \tag{5.9.23}$$

In Eqs. (5.9.23), ϕ_1 and ψ_1 are known functions of x.

We now repeat the above procedure by considering ϕ_3, ψ_3 to be the solution to the linear portion of Eq. (5.9.23):

$$\phi_3'' + \psi_3 - 2\phi_1\phi_3 = \phi_1^2$$

$$\psi_3'' + \phi_3 - \phi_1\psi_3 - \psi_1\phi_3 = \phi_1\psi_1 \tag{5.9.24}$$

We then consider ϕ_2, ψ_2 to be composed of ϕ_3, ψ_3 and some remainder ϕ_4, ψ_4,

$$\phi_2 = \phi_3 + \phi_4$$

$$\psi_2 = \psi_3 + \psi_4 \tag{5.9.26}$$

The equations for ϕ_4, ψ_4 are obtained by substituting Eqs. (5.9.26) in Eqs. (5.9.23) and employing Eqs. (5.9.24):

$$\phi_4'' + \psi_4 - 2\phi_1\phi_4 - 2\phi_3\phi_4 = \phi_3^2 + \phi_4^2$$

$$\psi_4'' + \phi_4 - \phi_1\phi_4 - \psi_1\phi_4 - \phi_3\psi_4 - \psi_3\phi_4 = \phi_3\psi_3 + \phi_4\psi_4 \tag{5.9.27}$$

In these equations ϕ_1, ϕ_3, ψ_1, ψ_3 are known functions. We proceed as before, solving the linear part of Eqs. (5.9.27) and proceed through n such steps until $|\phi_{2n-1}|$ and

$|\psi_{2n-1}|$ become small compared to the accumulated solutions

$$\phi \approx \phi_1 + \phi_3 + \cdots = \sum_{i=1}^{n} \phi_{2i-1}$$

$$\psi \approx \psi_1 + \psi_3 + \cdots = \sum_{i=1}^{n} \psi_{2i-1}$$

The main advantage of this method is that at each iteration after the first some features of the nonlinearities are included in the linear equations, while in Archer's method the particular form of the nonlinearity is relatively unimportant at a given solution step. The principal disadvantage is, of course, the increased complexity and logic involved. It can be seen from Eqs. (5.9.21), (5.9.24) and (5.9.27) that the linear equations to be solved change from step to step. However, the change is systematic and can be accounted for simply by updating the coefficient matrices in the finite difference scheme after an iteration. Other than the necessity of a few more computer instructions, the computational requirements for this method are not substantially more severe than in Archer's basic technique.

In the application of the above technique to Eqs. (5.9.6), (5.9.7), (5.9.18), and (5.9.19) convergence was obtained for all the combinations of parameters which were tried. The results of these computations are plotted on

Figures 25 through 43 where they are compared with the corresponding linear results for $\frac{a}{b} = 0.125$.

5.10 Comparison of Results for the Linear and Nonlinear Theories

Solutions for the stresses and displacements in the spinning shell fully clamped at the hub have been obtained by using the linear theory of Section 5.8 and the nonlinear theory of Section 5.9 with a grid of 36 points in the region $0.125 \leq X \leq 1$. We were required to impose the restriction of 36 grid points because of computational limitations. In some of the results some slight deviations from more exact results will be noted but the general agreement between comparable calculations is considered satisfactory. The results have been plotted in Figures 25 through 43 for four values of the shell geometry parameter $\lambda = 0.058$, 1.0, 3.0, and 7.0, which cover the case of a nearly flat shell to one with a fairly substantial curvature. For each value of λ we have obtained the linear solution and the nonlinear solution for five values of γ, which is the inertia loading parameter and scales the influence of the nonlinear terms. We have taken $\gamma = 0.01$, 0.3, 1.0, 6.0, and 120.0, which cover the range from very low to very high inertia loading. A value of $\gamma = 0.5$ would correspond to deflection of the shell to a flat plate in the nonlinear membrane shell theory (see Eq. (5.7.15)).

Figures 25 through 28 show the radial direct stress. For small λ it is seen that the stress distribution is nearly that of the spinning flat fully clamped disk. Furthermore, for $\lambda = 0.0580$ and $\lambda = 1.0$, practically no nonlinear effect occurs, as is seen by the fact that no variation with γ occurs, to within the accuracy of the graphs, and by the fact that the linear and nonlinear theories are in good agreement. For increasing λ we find a substantial variation of the solution with γ. For low γ the nonlinear and linear solutions nearly coincide while for large γ the solution is nearly that of the flat disk. In particular, for small γ and increasing λ, the characteristic decrease in importance of the radial stress is noted.

These phenomena are readily explainable on physical grounds. For small λ the shell is very nearly flat so that we expect the stress distribution to be near that of the flat disk. For higher λ curvature effects become important and for small γ a membrane stress distribution is approached. For increasing γ the disk progressively flattens out and the stress distribution alters radically until finally, for very high λ, the stress distribution is about the same as if the shell had been flat to start with. In this case the stress distribution required to overcome the initial curvature is small compared to the additional stress built up after the disk is essentially flat.

Figures 29 through 32 present computed results for the tangential direct stress. The same trends are seen here as were seen in the case of the radial direct stress. For small λ the stresses are those which occur in a flat disk and no nonlinear effect is possible. For high λ and small γ the linear membrane results are approached, and for increasing γ the nonlinear effects cause a radical variation in the stress. For very high γ the flat disk results again are approached.

Figures 33 through 40 are plots of the magnitude of the maximum bending stresses. The bending stresses were computed from the relations

$$\sigma_{rr_B} = \pm \frac{6 M_{rr}}{t^2}$$

$$\sigma_{\theta\theta_B} = \pm \frac{6 M_{\theta\theta}}{t^2}$$

The non-dimensional forms plotted in the figures, are

$$\sigma^*_{rr_B} = \frac{t\sigma_{rr_B}}{\rho\omega^2 b^2} = \frac{\sigma_{rr_B}}{\rho_0 \omega^2 b^2}$$

$$\sigma^{*}_{\theta\theta_B} = \frac{t\,\sigma_{\theta\theta_B}}{\rho\,\omega^2 b^2} = \frac{\sigma_{\theta\theta B}}{\rho_o\,\omega^2 b^2}$$

where we have introduced

$$\rho_o = m = \frac{\rho}{t} = \text{MATERIAL VOLUME DENSITY}$$

For $\lambda = 0.058$, the shell is nearly flat, and practically no bending stress is generated, and no nonlinear effect is seen. For $\lambda = 1.0$, the shell is still relatively flat so that very little nonlinear effect is seen until γ becomes very large and the disk is effectively flattened out. The bending stress shown for $\gamma = 120$ would be essentially the residual bending stress required to flatten the shell and would show almost no variation for further increases in γ. The parameter γ could also be lowered considerably before significant variation in the bending stress is seen. When λ is increased to 3 and above the nonlinear effects appear for relatively small values of γ. The effect of increasing γ is to decrease the bending stresses. For high γ the residual bending stress becomes small for the higher values of λ as the membrane effects become predominant. The characteristic initial increase in bending stress with increasing λ up to a point, and then decreasing bending stress with increasing λ is clearly evident.

Figures 41 through 43 present the results of the nonlinear theory for the bending slope. We have not included the case $\lambda = 0.058$ since the results for the bending slope are on the order of 10^{-7} and the single precision accuracy is open to question. For the case $\lambda = 1.0$ we have used a scale factor of 10^2 in order to maintain a uniformity in the figures. Further, note that we have employed the nondimensionalization

$$w^{*\prime} = \frac{d}{dx}\left(\frac{wEt}{\rho\omega^2 b^2 R}\right)$$

which can be written

$$w^{*\prime} = \frac{3+\nu}{2\gamma}\frac{d}{dx}\left(\frac{w}{h}\right)$$

A particularly interesting feature occurs for large λ when γ is large. Note when $\gamma = 120$ with $\lambda = 3.0$ or 7.0 the slope of the deflection curve is nearly linear. In this situation the disk has practically flattened out and the bending slope has become the negative of the original shell slope. This effect is not nearly as predominant for smaller λ . This implies h/t is small and both h small and t large would reduce the tendency to flatten out.

The result of primary interest from these computations is the strong nonlinear behavior of the solutions for increasing γ , particularly when λ is not very small. The substantial variation of the solutions with changing γ suggests that considerable care should be exercised if anything but the nonlinear theory is to be employed for the computation of the equilibrium configuration.

5.11 Conclusions Regarding the Appropriate Theory for Computation of the Equilibrium State

As will be seen in Chapter 6, the equations of motion of the spinning shallow spherical shell for free vibration about the equilibrium configuration depend strongly on the direct stresses and displacements which exist in the equilibrium state. For this reason particular care should be exercised in choosing the appropriate theory for the analysis of the steadily spinning shell.

By referring to Figures 20 through 43 and the preceeding analyses it is seen that for large λ a membrane theory is very nearly exact except for narrow regions near the boundaries where bending is important. The width of these regions decreases as λ increases.

For smaller values of λ , bending becomes important and a continuous transition to the flat plate results ($\lambda \to 0$) requires that it be considered.

Small values of the inertia loading parameter γ give rise to negligible nonlinear effects but these effects become increasingly more important as γ is increased. The effect of increasing γ is more predominant when λ is large.

It is concluded that any general purpose vibration analysis should be based on the nonlinear theory which includes bending effects. The use of a membrane theory, as employed by Johnson [8], is not appropriate, particularly for small λ . In addition, the effect of bending in the edge zones on the vibration characteristics should be examined in the membrane case. The use of a linear theory may well be adequate for most practical applications but its use would not substantially simplify the vibration analysis.

CHAPTER 6

THE EQUATIONS OF MOTION FOR THE FREE VIBRATIONS OF A SPINNING SHALLOW SPHERICAL SHELL

6.1 Introduction

The study of the transverse vibrations of a spinning shallow spherical shell presents two interesting features not found in analyses of non-spinning shells. The first, and most significant, variation introduced by the presence of spin is the generation of a pre-stressed equilibrium configuration about which the vibrations occur. This configuration has been studied in detail in Chapter 5. The second new feature which arises is the presence of a coriolis coupling between the three deflection components. These two additional considerations combine to produce a mathematical problem which is different and considerably more complex than the corresponding problem for the stationary shell. It is the purpose of this chapter to fully formulate the problem for numerical computation by an established technique. The available computation facility has proved to be inadequate to handle a problem of this size so that no numerical results have been obtained. Efforts are under way to pursue the matter further at another facility.

6.2 Basic Equations--Meridional, Tangential, and Normal Deflections

The differential equations which govern the small vibrations of a spinning shallow spherical shell about its equilibrium configuration were given in Eqs. (2.5.9), (2.5.10), and (2.5.11) for the case when the deflections are resolved in the meridional, tangential, and normal directions relative to the undeformed shell. They are repeated here for convenience with the now unstarred quantities understood to signify perturbation quantities and the quantities subscripted with "0" understood to represent equilibrium values:

$$\frac{\partial}{\partial r}(r\sigma_r) - \sigma_\theta + \frac{\partial \tau_{r\theta}}{\partial \theta} = mr\left[\ddot{u} - 2\omega\dot{v} - \left(u + \frac{\dot{\omega} r}{R}\right)\omega^2\right] \tag{6.2.1}$$

$$\frac{\partial}{\partial r}(r\tau_{r\theta}) + \frac{\partial \sigma_\theta}{\partial \theta} + \tau_{r\theta} = mr\left[\ddot{v} + 2\omega\left(\dot{u} + \dot{\omega}\frac{r}{R}\right) - \omega^2 v\right] \tag{6.2.2}$$

$$\frac{D}{t}\nabla^2\nabla^2 w - \frac{1}{r}\frac{\partial}{\partial r}\left(r\sigma_r w_{0_r} + r\sigma_{r_0} w_r\right)$$

$$-\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r}\sigma_{\theta_0} w_\theta + w_{0_r}\tau_{r\theta}\right) + \frac{1}{R}\left(\sigma_r + \sigma_\theta\right)$$

$$= -m\left[\ddot{w} - 2\omega\dot{v}\frac{r}{R} - \left(u + w\frac{r}{R}\right)\omega^2\frac{r}{R}\right] \tag{6.2.3}$$

In addition to the equations for dynamic equilibrium we must include the mid-surface strain-displacement relations for the perturbation quantities, here assumed small:

$$\epsilon_r = u_r + \frac{w}{R} + w_{0_r} w_r \tag{6.2.4}$$

$$\epsilon_\theta = \frac{u}{r} + \frac{v_\theta}{r} + \frac{w}{R} \tag{6.2.5}$$

$$\gamma_{r\theta} = \frac{u_\theta}{r} + v_r - \frac{v}{r} + \frac{w_{0_r} w_\theta}{r} \tag{6.2.6}$$

To complete the specification of the problem we state the boundary conditions for a shell of outer radius b with a fully clamped hub of radius a:

$$r = a: \quad u = 0$$

$$v = 0$$

$$w = 0$$

$$w_{,r} = 0 \tag{6.2.7}$$

$$r = b: \quad \sigma_r = 0$$

$$\tau_{r\theta} = 0$$

$$M_{rr} = -D\left(w_{,rr} + \frac{1}{r} w_{,r}\right) = 0$$

$$\text{KIRCHOFF COND: } \frac{\partial}{\partial r}\nabla^2 w + \frac{1-\nu}{r}\frac{\partial}{\partial r}\left(\frac{1}{r} w_{,\theta\theta}\right) = 0 \tag{6.2.8}$$

For the free vibration problem at hand we make use of the results of investigations of stationary shells to assume the following solution forms:

$$u(r,\theta,t) = U(r)\cos(n\theta + pt)$$

$$v(r,\theta,t) = V(r)\sin(n\theta + pt)$$

$$w(r,\theta,t) = W(r)\cos(n\theta + pt) \tag{6.2.9}$$

With this form for the displacements the strains are given by

$$\epsilon_r = \left(U_r + \frac{W}{R} + w_{o_r} W_r \right) \cos(n\theta + pt)$$

$$\epsilon_\theta = \left(\frac{U}{r} + \frac{nV}{r} + \frac{W}{R} \right) \cos(n\theta + pt)$$

$$\gamma_{r\theta} = \left(V_r - \frac{V}{r} - \frac{nU}{r} - n w_{o_r} \frac{W}{r} \right) \sin(n\theta + pt) \tag{6.2.10}$$

By utilizing the stress strain relations

$$\sigma_r = \frac{E}{1-\nu^2} \left[\epsilon_r + \nu \epsilon_\theta \right]$$

$$\sigma_\theta = \frac{E}{1-\nu^2} \left[\epsilon_\theta + \nu \epsilon_r \right]$$

$$\tau_{r\theta} = G \gamma_{r\theta} \tag{6.2.11}$$

we find that the stresses can be written

$$\sigma_r = S_r \cos(n\theta + pt)$$

$$\sigma_\theta = S_\theta \cos(n\theta + pt)$$

$$\tau_{r\theta} = S_{r\theta} \sin(n\theta + pt) \tag{6.2.12}$$

where

$$S_r = \frac{E}{1-\nu^2}\left\{ U_r + \frac{W}{R} + \omega_{o_r} W_r + \nu\left(\frac{U}{r} + \frac{W}{R} + n\frac{V}{r}\right)\right\}$$

$$S_\theta = \frac{E}{1-\nu^2}\left\{ \frac{U}{r} + \frac{W}{R} + n\frac{V}{r} + \nu\left(U_r + \frac{W}{R} + \omega_{o_r} W_r\right)\right\}$$

$$S_{r\theta} = \frac{E}{2(1+\nu)}\left\{ V_r - \frac{V}{r} - n\frac{U}{r} - n\omega_{o_r}\frac{W}{r}\right\} \qquad (6.2.13)$$

The equations of motion can be written

$$\frac{d}{dr}(r S_r) - S_\theta + n S_{r\theta} = -mr\left[p^2 U + 2\omega p V + \left(U + \frac{Wr}{R}\right)\omega^2\right] \qquad (6.2.14)$$

$$\frac{d}{dr}(r S_{r\theta}) - n S_\theta + S_{r\theta} = -mr\left[p^2 V + 2\omega p\left(U + \frac{Wr}{R}\right) + \omega^2 V\right] \qquad (6.2.15)$$

$$\frac{D}{t} L^4 W - \frac{1}{r}\frac{d}{dr}\left(r\,\omega_{o_r} S_r + r\sigma_{r_o} W_r\right)$$
$$+ \frac{1}{r}\left(\frac{n^2}{r}\sigma_{\theta_o} W - n\,\omega_{o_r} S_{r\theta}\right) + \frac{1}{R}(S_r + S_\theta)$$
$$= m\left[p^2 W + 2\omega p V\frac{r}{R} + \left(U + \frac{Wr}{R}\right)\omega^2\frac{r}{R}\right] \qquad (6.2.16)$$

where

$$L^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2}$$

By using Eqs. (6.2.13), Eqs. (6.2.14), (6.2.15), and (6.2.16) can be written

$$r U_{rr} + r w_{o_r} W_{rr} + U_r + \frac{n(1+\nu)}{2} V_r + \left[(1+\nu)\frac{r}{R} + (1-\nu) w_{o_r} + r w_{o_{rr}}\right] W_r$$

$$- \left[1 + \frac{n^2}{2}(1-\nu)\right]\frac{U}{r} - \frac{n(3-\nu)}{2}\frac{V}{r} - \left[\frac{n^2}{2}(1-\nu) w_{o_r}\right]\frac{W}{r}$$

$$+ \frac{m(1-\nu^2)\omega^2}{E} r \left\{\left[1+\left(\frac{f}{\omega}\right)^2\right] U + 2\left(\frac{f}{\omega}\right) V + \frac{r}{R} W\right\} = 0$$

(6.2.17)

$$\frac{1-\nu}{2} r V_{rr} - \frac{n(1+\nu)}{2} U_r + \frac{1-\nu}{2} V_r - \frac{n(1+\nu)}{2} w_{o_r} W_r - \frac{n(3-\nu)}{2}\frac{U}{r}$$

$$- \left(n^2 + \frac{1-\nu}{2}\right)\frac{V}{r} - n\left[\frac{1+\nu}{R} + \frac{1-\nu}{2}\left(w_{o_{rr}} + \frac{w_{o_r}}{r}\right)\right] W$$

$$+ \frac{m(1-\nu^2)\omega^2}{E} r \left\{\left[1+\left(\frac{f}{\omega}\right)^2\right] V + 2\left(\frac{f}{\omega}\right)\left(U + W\frac{r}{R}\right)\right\} = 0$$

(6.2.18)

$$\frac{D}{t} L^4 W - \frac{1}{r}\frac{d}{dr}\left(r w_{o_r} S_r + r \sigma_{r_o} W_r\right)$$

$$+ \frac{1}{r}\left(\frac{n^2}{r}\sigma_{\theta_o} W - n w_{o_r} S_{r\theta}\right) + \frac{1}{R}\left(S_r + S_\theta\right)$$

$$= m\omega^2\left[\left(\frac{f}{\omega}\right)^2 W + 2\left(\frac{f}{\omega}\right) V \frac{r}{R} + \left(U + W\frac{r}{R}\right)\frac{r}{R}\right] \quad (6.2.19)$$

It proves convenient to cast Eqs. (6.2.17), (6.2.18) and (6.2.19) in non-dimensional form by introducing

$$x = \frac{r}{b}$$

$$U = U_o u(x)$$

$$V = V_o v(x)$$

$$W = W_o w(x) \quad (6.2.20)$$

$$U_o = V_o = W_o\left(\frac{b}{R}\right)$$

Further, from Chapter 5, Eqs. (5.9.11), (5.9.12), and (5.9.13) we have in the equilibrium configuration

$$\sigma_{r_0} = \frac{N_{rr}}{t} = \frac{(3+\nu)\rho\omega^2 b^2}{t}\,\frac{f(x)}{x}$$

$$= \frac{(3+\nu)\rho\omega^2 R^2}{Et} E\left(\frac{b}{R}\right)^2 \frac{f(x)}{x} \tag{6.2.21}$$

$$\sigma_{\theta_0} = \frac{N_{\theta\theta}}{t} = \frac{(3+\nu)\rho\omega^2 R^2}{Et} E\left(\frac{b}{R}\right)^2 \left[f'(x) + \frac{x^2}{3+\nu}\right]$$

$$= \frac{(3+\nu)\rho\omega^2 R^2}{Et} E\left(\frac{b}{R}\right)^2 h(x) \tag{6.2.22}$$

$$\frac{R}{b^2}\frac{dw_0}{dx} = \frac{R}{b^2}\,\frac{(3+\nu)\rho\omega^2 R^2}{E\sqrt{12(1-\nu^2)}}\,\lambda^2 g(x)$$

$$= \frac{(3+\nu)\rho\omega^2 R^2}{Et}\, g(x) \tag{6.2.23}$$

With the definition

$$\gamma = \frac{(3+\nu)\rho\omega^2 R^2}{Et} = \frac{(3+\nu)\, m\omega^2 R^2}{E} \tag{6.2.24}$$

we write

$$\sigma_{r_0} = \gamma E \left(\frac{b}{R}\right)^2 \frac{f(x)}{x} \tag{6.2.25}$$

$$\sigma_{\theta_0} = \gamma E \left(\frac{b}{R}\right)^2 h(x) \tag{6.2.26}$$

$$\frac{R}{b^2}\frac{dw_0}{dx} = \gamma' g(x) \tag{6.2.27}$$

By introducing Eqs. (6.2.20) into Eqs. (6.2.13) we find that the stresses in terms of the displacements can be written

$$S_r = \frac{E}{1-\nu^2}\frac{W_0}{R}\left\{u_x + w + \left(\frac{R}{b^2}w_{0x}\right)w_x + \nu\left(\frac{u}{x} + w + n\frac{v}{x}\right)\right\}$$

$$S_\theta = \frac{E}{1-\nu^2}\frac{W_0}{R}\left\{\frac{u}{x} + w + \frac{nv}{x} + \nu\left[u_x + w + \left(\frac{R}{b^2}w_{0x}\right)w_x\right]\right\}$$

$$S_{r\theta} = \frac{1-\nu}{2}\frac{E}{1-\nu^2}\frac{W_0}{R}\left\{v_x - \frac{v}{x} - \frac{nu}{x} - n\left(\frac{R}{b^2}w_{0x}\right)\frac{w}{x}\right\} \tag{6.2.28}$$

With Eq. (6.2.27) we can then obtain

$$S_r = \frac{E}{1-\nu^2}\frac{W_0}{R}\left\{u_x + w + \gamma' g(x) w_x + \nu\left(\frac{u}{x} + w + \frac{nv}{x}\right)\right\}$$

$$S_\theta = \frac{E}{1-\nu^2}\frac{W_0}{R}\left\{\frac{u}{x} + w + \frac{nv}{x} + \nu\left[u_x + w + \gamma' g(x) w_x\right]\right\}$$

$$S_{r\theta} = \frac{1-\nu}{2}\frac{E}{1-\nu^2}\frac{W_0}{R}\left\{v_x - \frac{v}{x} - \frac{nu}{x} - n\gamma' g(x)\frac{w}{x}\right\} \tag{6.2.29}$$

Equations (6.2.29) can then be written in the convenient form

$$S_r = \frac{E}{1-\nu^2}\frac{W_0}{R}s_r$$

$$S_\theta = \frac{E}{1-\nu^2}\frac{W_0}{R}s_\theta$$

$$S_{r\theta} = \frac{1-\nu}{2}\frac{E}{1-\nu^2}\frac{W_0}{R}s_{r\theta}$$

where s_r , s_θ , $s_{r\theta}$ are defined by the bracketed terms in Eqs. (6.2.29):

$$s_r = u_x + w + \gamma g(x)\, w_x + \nu\left(\frac{u}{x} + w + n\frac{v}{x}\right)$$

$$s_\theta = \frac{u}{x} + w + n\frac{v}{x} + \nu\left(u_x + w + \gamma g(x)\, w_x\right)$$

$$s_{r\theta} = v_x - \frac{v}{x} - n\frac{u}{x} - n\gamma g(x)\frac{w}{x} \qquad (6.2.30)$$

By utilizing the relations of Eqs. (6.2.20) through (6.2.30) we obtain the non-dimensional equations of motion

$$x u_{xx} + \gamma x g(x) w_{xx} + u_x + \frac{n(1+\nu)}{2} v_x + \Big[(1+\nu)x + (1-\nu)\gamma g(x)$$

$$+ \gamma x g_x(x)\Big] w_x - \left[1 + \frac{n^2}{2}(1-\nu)\right]\frac{u}{x} - \frac{n(3-\nu)}{2}\frac{v}{x} - \frac{\gamma n^2}{2}(1-\nu) g(x)\frac{w}{x}$$

$$+ \frac{1-\nu^2}{3+\nu}\gamma\left(\frac{b}{R}\right)^2 x\left\{\left[1+\left(\frac{\ell}{\omega}\right)^2\right]u + 2\left(\frac{\ell}{\omega}\right)v + xw\right\} = 0 \qquad (6.2.31)$$

$$\frac{1-\nu}{2} x v_{xx} - \frac{n(1+\nu)}{2} u_x + \frac{1-\nu}{2} v_x - \frac{\gamma(1+\nu)}{2} g(x) w_x - \frac{n(3-\nu)}{2}\frac{u}{x}$$

$$-\left(n^2 + \frac{(1-\nu)}{2}\right)\frac{v}{x} - n\left\{(1+\nu) + \frac{1-\nu}{2}\left[\gamma g_x(x) + \frac{\gamma}{x} g(x)\right]\right\} w$$

$$+ \frac{1-\nu^2}{3+\nu}\gamma\left(\frac{b}{R}\right)^2 x\left\{\left[1+\left(\frac{\ell}{\omega}\right)^2\right]v + 2\left(\frac{\ell}{\omega}\right)(u+xw)\right\} = 0 \qquad (6.2.32)$$

$$L_x^4 w - \frac{\lambda^4\gamma}{x}\frac{d}{dx}\left[\frac{1}{1-\nu^2} x g(x) s_r + f(x) w_x\right]$$

$$+ \frac{\lambda^4\gamma}{x}\left[n^2\frac{h(x)}{x} w - \frac{n\gamma}{2(1+\nu)} g(x) s_{r\theta}\right] + \frac{\lambda^4}{1-\nu^2}(s_r + s_\theta)$$

$$= \frac{\lambda^4\gamma}{3+\nu}\left[\left(\frac{\ell}{\omega}\right)^2 w + 2\left(\frac{\ell}{\omega}\right)\left(\frac{b}{R}\right)^2 x v + \left(\frac{b}{R}\right)^2 x(u+xw)\right] \qquad (6.2.33)$$

The corresponding non-dimensional boundary conditions are

$$x = a/b: \qquad \begin{aligned} u &= 0 \\ v &= 0 \\ w &= 0 \\ w_x &= 0 \end{aligned} \tag{6.2.34}$$

$x = 1$:

$$S_r = u_x + w + \gamma g(x) w_x + \nu\left(\frac{u}{x} + w + \frac{nv}{x}\right) = 0$$

$$S_{r\theta} = v_x - \frac{v}{x} - \frac{nu}{x} - n\gamma g(x)\frac{w}{x} = 0$$

$$M_{rr} = -D\left(w_{xx} + \frac{w_x}{x}\right) = 0$$

$$\frac{d}{dx} L_x^2 w - n^2 \frac{1-\nu}{x}\frac{d}{dx}\left(\frac{w}{x}\right) = 0 \tag{6.2.35}$$

It is seen that the non-dimensional equations of motion depend on four parameters:

$$\gamma = \frac{(3+\nu) m \omega^2 R^2}{E}$$

$$\lambda^4 = \frac{Etb^4}{DR^2} \approx 2\sqrt[4]{3(1-\nu^2)}\sqrt{h/t}$$

ν = POISSON'S RATIO

$\frac{b}{R} \approx 2\frac{h}{b}$ = SHELL SHALLOWNESS PARAMETER

In addition, the boundary conditions require the specification of the ratio of the hub radius to the disk radius, $\frac{a}{b}$. It is observed that the dynamic problem requires the specification of one additional parameter as compared to the equilibrium problem. It can be seen by referring to Eqs. (6.2.31), (6.2.32), and (6.2.33) that the extra parameter, $\frac{b}{R}$, scales the relative importance of the inertia terms in Eqs. (6.2.31) and (6.2.32) and the Coriolis and centripetal acceleration terms in Eq. (6.2.33). Since the shallow shell theory used here implies the neglect of terms of the order of $\left(b/R\right)^2$ it is concluded that terms of this order in the equations of motion can also be dropped, provided that only the transverse motions are of interest. This is in agreement with the results of Reissner [22] regarding the neglect of longitudinal inertia for primarily transverse vibrations of shallow shells and extends the conclusion to include the coriolis and centripetal coupling terms in the equation for transverse motion. As shown by Reissner, if advantage is taken of the neglect of these higher order terms the governing equations can be simplified by the introduction of a stress function which identically satisfies Eqs. (6.2.1) and (6.2.2) in the case when the right hand sides are zero. The equation satisfied by the stress function is found by employing a suitable compatibility equation.

6.3 Basic Equations--Radial, Tangential, and Axial Deflections

For some purposes it may prove more convenient to have the governing equations resolved in the radial, tangential, and axial directions as was done by Johnson [8] in the membrane case. The equations for this case were originally given by Eqs. (2.6.15), (2.6.16), and (2.6.19) and are repeated here with the stars and bars suppressed

$$\frac{\partial}{\partial r}(r\sigma_r) - \sigma_\theta + \frac{\partial \tau_{r\theta}}{\partial \theta} = mr\left[\ddot{u} - 2\omega\dot{v} - u\omega^2\right] \tag{6.3.1}$$

$$\frac{\partial}{\partial r}(r\tau_{r\theta}) + \frac{\partial \sigma_\theta}{\partial \theta} + \tau_{r\theta} = mr\left[\ddot{v} + 2\omega\dot{u} - v\omega^2\right] \tag{6.3.2}$$

$$\frac{D}{t}\nabla^2\nabla^2 w - \frac{1}{r}\frac{\partial}{\partial r}\left\{r\left(-\frac{r}{R} + w_{0_r}\right)\sigma_r + r\sigma_{r_0} w_r\right\}$$
$$-\frac{1}{r}\frac{\partial}{\partial \theta}\left\{\left(-\frac{r}{R} + w_{0_r}\right)\tau_{r\theta} + \frac{1}{r}\sigma_{\theta_0} w_\theta\right\} = -m\ddot{w} \tag{6.3.3}$$

The corresponding strain-displacement relations for the small perturbation quantities are

$$\epsilon_r = u_r + w_{0_r} w_r - \frac{r}{R} w_r \tag{6.3.4}$$

$$\epsilon_\theta = \frac{v_\theta}{r} + \frac{u}{r} \tag{6.3.5}$$

$$\gamma_{r\theta} = \frac{u_\theta}{r} + v_r - \frac{v}{r} + \frac{w_{0_r} w_\theta}{r} - \frac{w_\theta}{R} \tag{6.3.6}$$

The boundary conditions for the case of a fully clamped hub of radius a and a free outer edge will be given by Eqs. (6.2.7) and (6.2.8).

By utilizing the wave type solutions given by Eqs. (6.2.9) the strain displacement relations can be written

$$\epsilon_r = \left(U_r + \omega_{o_r} W_r - \frac{r}{R} W_r \right) \cos(n\theta + pt) \tag{6.3.7}$$

$$\epsilon_\theta = \left(\frac{U}{r} + n\frac{V}{r} \right) \cos(n\theta + pt) \tag{6.3.8}$$

$$\gamma_{r\theta} = \left(V_r - n\frac{U}{r} - \frac{V}{r} + n\frac{W}{R} - n\omega_{o_r}\frac{W}{r} \right) \sin(n\theta + pt) \tag{6.3.9}$$

The stress-strain relations can then be employed to yield

$$\begin{aligned} \sigma_r &= S_r \cos(n\theta + pt) \\ \sigma_\theta &= S_\theta \cos(n\theta + pt) \\ \tau_{r\theta} &= S_{r\theta} \sin(n\theta + pt) \end{aligned} \tag{6.3.10}$$

where

$$S_r = \frac{E}{1-\nu^2}\left\{U_r + \omega_{o_r} W_r - \frac{r}{R} W_r + \nu\left(\frac{U}{r} + \frac{nV}{r}\right)\right\}$$

$$S_\theta = \frac{E}{1-\nu^2}\left\{\frac{U}{r} + \frac{nV}{r} + \nu\left(U_r + \omega_{o_r} W_r - \frac{r}{R} W_r\right)\right\}$$

$$S_{r\theta} = G\left(V_r - n\frac{U}{r} - \frac{V}{r} + n\frac{W}{R} - n\omega_{o_r}\frac{W}{r}\right) \tag{6.3.11}$$

In terms of the assumed form of solution the equations of motion can be written

$$\frac{d}{dr}(rS_r) - S_\theta + nS_{r\theta} = -mr\left[(p^2+\omega^2)U + 2\omega p V\right] \tag{6.3.12}$$

$$\frac{d}{dr}(rS_{r\theta}) - nS_\theta + S_{r\theta} = -mr\left[(p^2+\omega^2)V + 2\omega p U\right] \tag{6.3.13}$$

$$\frac{D}{t} L^4 W - \frac{1}{r}\frac{d}{dr}\left\{r\left(-\frac{r}{R} + \omega_{o_r}\right)S_r + r\sigma_{r_o} W_r\right\}$$
$$+ \frac{n^2}{r^2}\sigma_{\theta_o} W - \frac{n}{r}\left(-\frac{r}{R} + \omega_{o_r}\right)S_{r\theta} = m p^2 \omega \tag{6.3.14}$$

where

$$L^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2}$$

The equations of motion in terms of displacements become

$$r U_{rr} + r(\omega_{o_r} - r/R) W_{rr} + U_r + \frac{n(1+\nu)}{2} V_r + [(1-\nu)(\omega_{o_r} - r/R) + r(\omega_{o_r} - r/R)_r] W_r$$
$$- [1 + n^2 \frac{(1-\nu)}{2}] \frac{U}{r} - n\frac{(3-\nu)}{2}\frac{V}{r} - \frac{n^2(1-\nu)}{2}(\omega_{o_r} - \frac{r}{R})\frac{W}{R}$$
$$+ \frac{(1-\nu^2) m\omega^2 r}{E}\left\{[1+(\frac{\ell}{\omega})^2] U + 2(\frac{\ell}{\omega}) V\right\} = 0 \tag{6.3.15}$$

$$(1-\nu) r V_{rr} - \frac{n(1+\nu)}{2} U_r + \frac{1-\nu}{2} V_r - \frac{n(1+\nu)}{2}(\omega_{o_r} - \frac{r}{R}) W_r - \frac{n(3-\nu)}{2}\frac{U}{r}$$
$$-(n^2 + \frac{1-\nu}{2})\frac{V}{r} - \frac{n(1-\nu)}{2}[r(\omega_{o_r} - r/R)_r + (\omega_{o_r} - r/R)]\frac{W}{r}$$
$$+ \frac{(1-\nu^2) m\omega^2 r}{E}\left\{[1+(\frac{\ell}{\omega})^2] V + 2(\frac{\ell}{\omega}) U\right\} = 0 \tag{6.3.16}$$

$$\frac{D}{t} L^4 W - \frac{1}{r}\frac{d}{dr}\left\{r(-\frac{r}{R} + \omega_{o_r}) S_r + r\sigma_{r_o} W_r\right\}$$
$$+ \frac{n^2}{r^2}\sigma_{\theta_o} W - \frac{n}{r}(-\frac{r}{R} + \omega_{o_r}) S_{r\theta} = m\omega^2 (\frac{\ell}{\omega})^2 W \tag{6.3.17}$$

where S_r and $S_{r\theta}$ are defined by Eqs. (6.3.11). We obtain non-dimensional equations in the same way as in Section 6.2 by employing Eqs. (6.2.20) through (6.2.27) together with the appropriate non-dimensional form of the stress-displacement relations for this case:

$$S_r = \frac{E}{1-\nu^2}\frac{W_o}{R}\left\{u_x + [\gamma g(x) - x] w_x + \nu(\frac{u}{x} + \frac{n v}{x})\right\}$$

$$S_\theta = \frac{E}{1-\nu^2}\frac{W_o}{R}\left\{\frac{u}{x} + \frac{n v}{x} + \nu(u_x + [\gamma g(x) - x] w_x)\right\}$$

$$S_{r\theta} = \frac{1-\nu}{2}\frac{E}{1-\nu^2}\frac{W_o}{R}\left\{v_x - \frac{n u}{x} - \frac{v}{x} + n[x - \gamma g(x)]\frac{w}{x}\right\}$$

By introducing the non-dimensional stresses s_r , s_θ , $s_{r\theta}$, defined by

$$s_r = u_x + [\gamma g(x) - x]\,\omega_x + \nu\left(\frac{u}{x} + \frac{n v}{x}\right)$$

$$s_\theta = \frac{u}{x} + \frac{n v}{x} + \nu\left(u_x + [\gamma g(x) - x]\,\omega_x\right)$$

$$s_{r\theta} = v_x - \frac{n u_x}{x} - \frac{v}{x} + n[x - \gamma g(x)]\,\frac{\omega}{x} \qquad (6.3.18)$$

we have for the dimensional stresses

$$S_r = \frac{E}{1-\nu^2}\,\frac{W_0}{R}\,s_r$$

$$S_\theta = \frac{E}{1-\nu^2}\,\frac{W_0}{R}\,s_\theta$$

$$S_{r\theta} = \frac{1-\nu}{2}\,\frac{E}{1-\nu^2}\,\frac{W_0}{R}\,s_{r\theta}$$

The non-dimensional equations of motion for this case are

$$x u_{xx} + x[\gamma g(x) - x]\,\omega_{xx} + u_x + \frac{n(1+\nu)}{2} v_x + \left[(1-\nu)(\gamma g(x) - x) + x(\gamma g(x) - x)_x\right]\omega_x$$
$$-\left[1 + n^2\frac{(1-\nu)}{2}\right]\frac{u}{x} - \frac{n(3-\nu)}{2}\frac{v}{x} - \frac{n^2(1-\nu)}{2}[\gamma g(x) - x]\,\frac{\omega}{x}$$
$$+ \frac{1-\nu^2}{3+\nu}\,\gamma\left(\frac{b}{R}\right)^2 x\left\{\left[1 + \left(\frac{\ell}{\omega}\right)^2\right]u + 2\left(\frac{\ell}{\omega}\right)v\right\} = 0 \qquad (6.3.19)$$

$$\frac{1-\nu}{2} x v_{xx} - \frac{n(1+\nu)}{2} u_x + \frac{1-\nu}{2} v_x - \frac{n(1+\nu)}{2}[\gamma g(x) - x]\,\omega_x$$
$$- \frac{n(3-\nu)}{2}\frac{u}{x} - \left(n^2 + \frac{1-\nu}{2}\right)\frac{v}{x} - \frac{n(1-\nu)}{2}\left[x(\gamma g(x) - x)_x\right.$$
$$\left. + (\gamma g(x) - x)\right]\frac{\omega}{x} + \frac{1-\nu^2}{3+\nu}\,\gamma\left(\frac{b}{R}\right)^2 x\left\{\left[1 + \left(\frac{\ell}{\omega}\right)^2\right]v + 2\left(\frac{\ell}{\omega}\right)u\right\} = 0 \qquad (6.3.20)$$

$$L_x^4\,\omega - \frac{\lambda^4}{x}\frac{d}{dx}\left\{\frac{1}{1-\nu^2}\,x[\gamma g(x) - x]\,s_r + \gamma f(x)\,\omega_x\right\}$$
$$+ \lambda^4 \gamma \frac{n^2}{x^2} h(x)\,\omega - \frac{\lambda^4}{2(1+\nu)}\frac{n}{x}[\gamma g(x) - x]\,s_{r\theta} = \frac{\lambda^4\gamma}{3+\nu}\left(\frac{\ell}{\omega}\right)^2\omega \qquad (6.3.21)$$

Equations (6.2.34) and (6.2.35) can also be employed to specify the boundary conditions for this case except that the appropriate definitions of S_r and $S_{r\theta}$ must be employed in Eqs. (6.2.35). As in the case of meridional, tangential, and normal deflections we find that the differential equations and boundary conditions depend on five parameters. In addition, we once again note the relative unimportance of the longitudinal inertia terms in the membrane equilibrium equations.

There does not appear to be any particular advantage in choosing one or the other of the two forms of the governing equations of motion for purposes of numerical computations. Although the form valid for radial, tangential, and axial deflections appears to be slightly simpler in the bending equation, a few manipulations can be employed to reduce the other equations to a nearly equivalent form.

6.4 Numerical Computation of the Natural Frequencies and Mode Shapes

The method which is proposed here for the numerical computation of the natural frequencies and mode shapes is an adaptation of a technique reported by Zarghamee and Robinson [26]. Reduced to its basic concepts this method is an extension of the technique used in Chapters 4 and 5 to determine the natural frequencies for flat membrane disks.

Since the problem is linear we know that a solution can be obtained as the sum of a number of independent fundamental solutions. In particular, we choose to define our fundamental solutions as those solutions which satisfy the boundary conditions at one boundary but which have unspecified values at the other boundary. By superposition of these solutions with suitable arbitrary constants we obtain a general solution whose constants can be determined by the requirement that the remaining boundary conditions be satisfied. This procedure leads to a set of homogeneous algebraic equations for the undetermined constants which can only be satisfied in a non-trivial way if the determinant of the coefficients vanishes. The determinant of the coefficients is formed from the fundamental solutions and certain of their derivatives evaluated at the boundary in question. Since these solutions depend on the undetermined natural frequency we find that the problem reduces to that of finding values of the natural frequency for which the determinant of coefficients vanishes.

By referring to Chapters 4 and 5, which discuss the vibrations of spinning membrane disks, it can be verified that the fundamental solution was taken as that solution which satisfies the finiteness condition at the outer edge of the disk. The vanishing of the deflection at the hub leads to the condition that the fundamental solution itself vanish at the hub. It is seen that the determinant of coefficients in this case reduces to a single element. The disk vibration analyses led to exceptionally simple eigenvalue problems since the boundary conditions were simple and the form of the fundamental solutions was known apriori.

The problem at hand is conceptually the same but practically considerably more difficult. The boundary conditions, of course, are more involved, requiring four conditions at each boundary. However, the biggest complication is that the form of the fundamental solutions is not known apriori, but rather, must be determined by numerical integration of the equations of motion. The eigenvalue equation requires the determination of the zeroes of the determinant of coefficients whose elements are obtained by numerically integrating the equations of motion.

In the present problem we will identify four linearly independent fundamental solutions which satisfy initial value problems defined as follows:

$$
\begin{array}{lll}
(1) \quad O(u, v, w) = 0 & & \\
u(a/b) = 0 & u_x(a/b) = 1 & \\
v(a/b) = 0 & v_x(a/b) = 0 & \\
w(a/b) = 0 & w_{xx}(a/b) = 0 & \\
w_x(a/b) = 0 & w_{xxx}(a/b) = 0 & (6.4.1)
\end{array}
$$

Yields fundamental solution $u_1(x)$, $v_1(x)$, $w_1(x)$

(2) $O(u, v, w) = 0$

$$u(a/b) = 0 \qquad u_x(a/b) = 0$$

$$v(a/b) = 0 \qquad v_x(a/b) = 1$$

$$w(a/b) = 0 \qquad w_{xx}(a/b) = 0$$

$$w_x(a/b) = 0 \qquad w_{xxx}(a/b) = 0 \tag{6.4.2}$$

Yields fundamental solution $u_2(x), v_2(x), w_2(x)$

(3) $O(u, v, w) = 0$

$$u(a/b) = 0 \qquad u_x(a/b) = 0$$

$$v(a/b) = 0 \qquad v_x(a/b) = 0$$

$$w(a/b) = 0 \qquad w_{xx}(a/b) = 1$$

$$w_x(a/b) = 0 \qquad w_{xxx}(a/b) = 0 \tag{6.4.3}$$

Yields fundamental solution $u_3(x), v_3(x), w_3(x)$

(4) $O(u, v, w) = 0$

$$u(a/b) = 0 \qquad u_x(a/b) = 0$$

$$v(a/b) = 0 \qquad v_x(a/b) = 0$$

$$w(a/b) = 0 \qquad w_{xx}(a/b) = 0$$

$$w_x(a/b) = 0 \qquad w_{xxx}(a/b) = 1 \tag{6.4.4}$$

Yields fundamental solution $u_4(x), v_4 \ , w_4(x)$

In the above described initial value problems the linear operator $O(u, v, w) = 0$ represents the differential equations from Eqs. (6.2.31), (6.2.32), and (6.2.33) or Eqs. (6.3.19), (6.3.20), and (6.3.21). This operator is also a function of the unknown parameter containing the natural frequency of vibration.

If we denote a fundamental solution vector by

$$\underline{\Phi}_i(x) = \begin{Bmatrix} u_i(x) \\ v_i(x) \\ w_i(x) \end{Bmatrix} \qquad i = 1, 2, 3, 4 \tag{6.4.5}$$

then by superposition we obtain a general solution

$$\underline{\Phi}(x) = \sum_{i=1}^{4} A_i \, \underline{\Phi}_i(x) \tag{6.4.6}$$

of the differential equations

$$O(\underline{\Phi}) = 0 \tag{6.4.7}$$

This general solution has four arbitrary constants to be determined so that the free edge conditions are satisfied. The free edge conditions were given by Eqs. (6.2.35) for the case of meridional, tangential, and normal deflections and by these same equations with a suitable modification of the definition of S_r and $S_{r\theta}$ in the case of radial, tangential, and axial deflections. In either case, the boundary conditions can be written in operator form

$$B(\underline{\Phi})_{x=1} = 0$$

(6.4.8)

Where B is the linear operator appropriate to the particular boundary conditions used. The boundary conditions at the free edge are seen to lead to four simultaneous homogeneous algebraic equations for the A_i

$$\sum_{i=1}^{4} A_i \, B(\underline{\Phi}_i)_{x=1} = 0$$

(6.4.9)

To emphasize the dependence of the fundamental solutions on the frequency parameter $\left(\frac{\ell}{\omega}\right)$ we write the algebraic equations

$$\sum_{i=1}^{4} A_i \, B\left[\underline{\Phi}_i\left(\frac{\ell}{\omega}\right)\right]_{x=1} = 0$$

(6.4.10)

These equations will have a non-trivial solution only if the determinant of the coefficients of the A_i vanishes. This determinant is seen to be a function of the frequency parameter $\left(\frac{\ell}{\omega}\right)$ so that we will have for a non-trivial solution

$$\Delta\left(\frac{p}{\omega}\right) = 0$$

(6.4.11)

where $\Delta\left(\frac{p}{\omega}\right)$ is the determinant of the coefficients of the set of algebraic equations in Eqs. (6.4.10).

The eigenvalue problem is seen to be that of finding values of (p/ω) for which Eq. (6.4.11) is satisfied. The method of approach for solving the eigenvalue equation is to evaluate the determinant for a sequence of values of (p/ω) until a sign change in $\Delta(p/\omega)$ is noted. At this point a suitable iteration scheme is employed to obtain an accurate estimate of the eigenvalue. This scheme is the same as the one used in the disk vibration problems but is complicated by the fact that each evaluation of the determinant requires four numerical integrations of the governing differential equations.

Since we are dealing here with shallow shells we do not anticipate the requirement that certain fundamental solutions be suppressed to prevent unbounded growth as discussed by Zarghamee and Robinson. However, as these authors also point out, this feature may have to be available if extremely thin shells are to be satisfactorily treated, and even this capability may be insufficient if the limiting case of a true membrane shell is approached.

We currently have available the sub-programs required for the construction of a digital computer program to carry out the above described computations. They include a program to compute and tabulate the equilibrium stress and displacement distribution, a numerical integration scheme for determining the fundamental solutions for specified value of (p/ω), a determinant evaluation procedure for obtaining $\Delta(p/\omega)$, and an iteration procedure for obtaining the zeroes of $\Delta(p/\omega)$. In addition, these sub-programs could be used to compute the mode shapes for a given frequency if required.

We have found the available IBM 1620 Computer Facility to be too limited to handle problems of this size. While most of the sub-programs have been checked out on this facility no results have been obtained for the complete problem. It is hoped that within the near future these computations can be completed at a larger computation facility.

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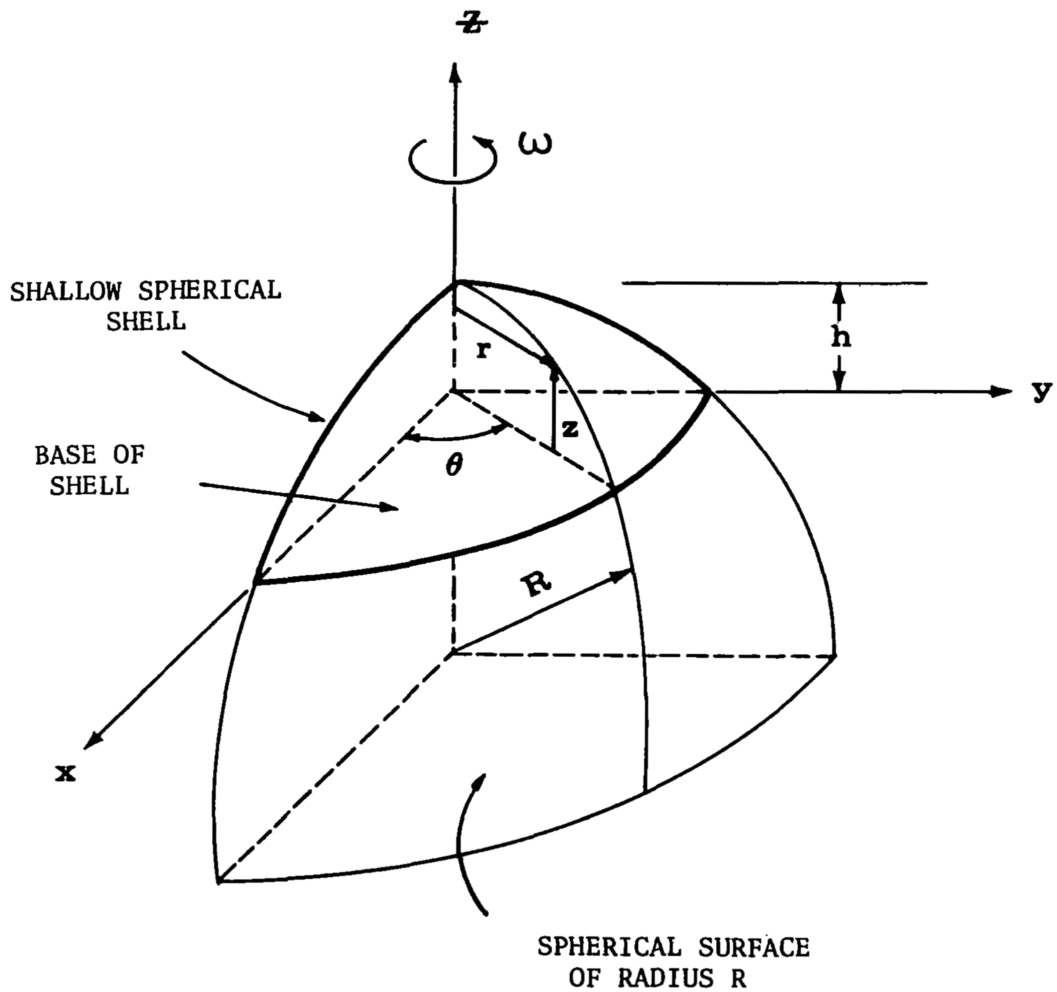


Figure 1. Geometry of the Spinning Shallow Spherical Shell.

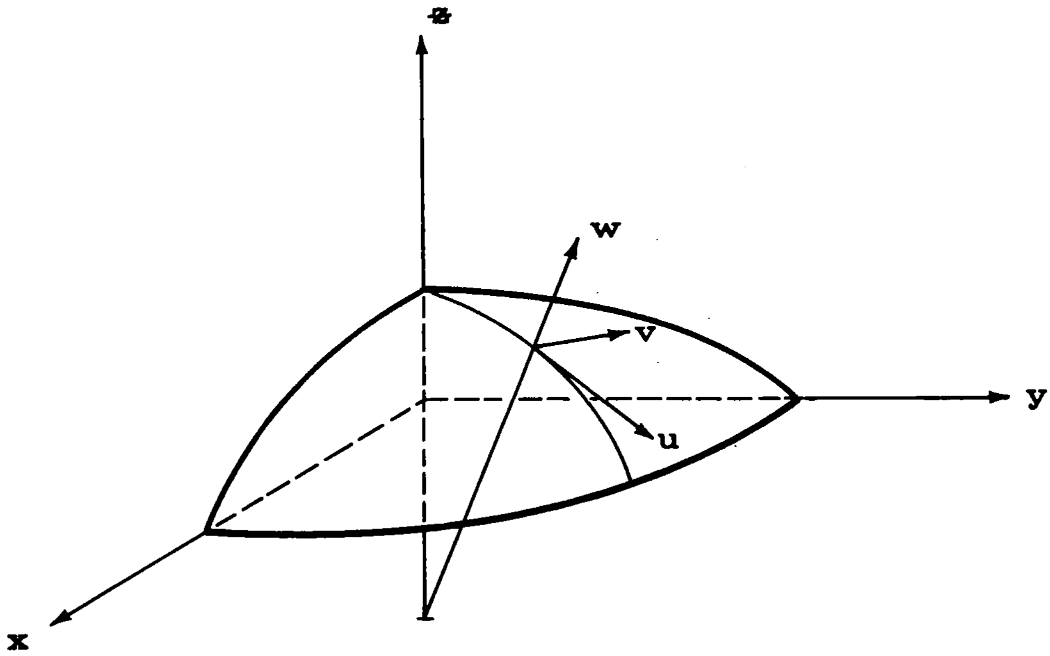


Figure 2. Coordinate System in which Displacements are Measured in the Meridional, Tangential, and Normal Directions.

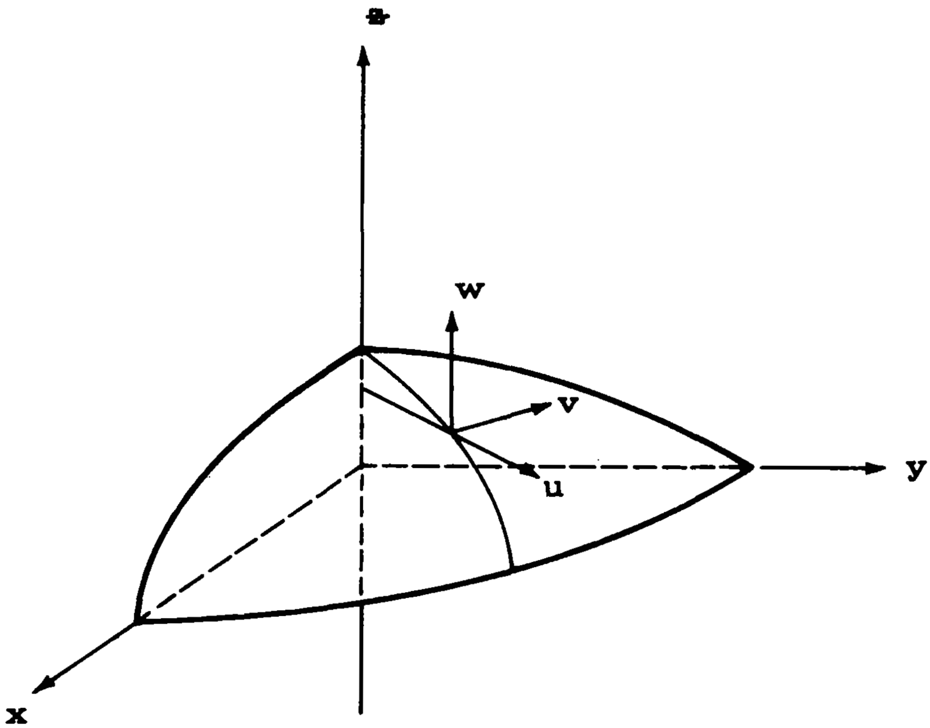
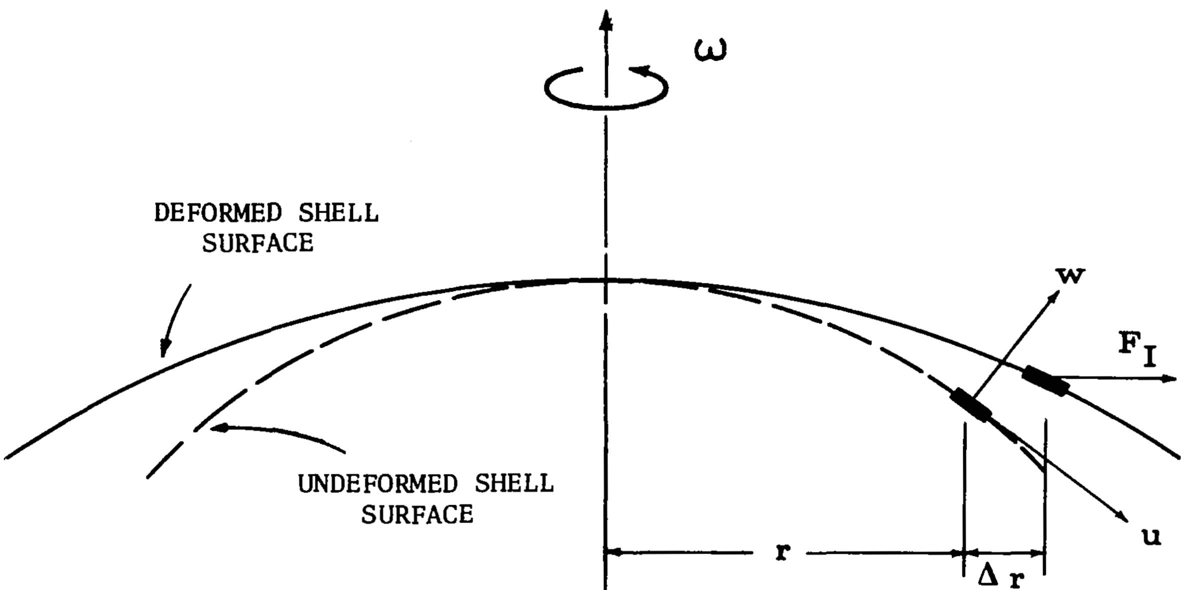


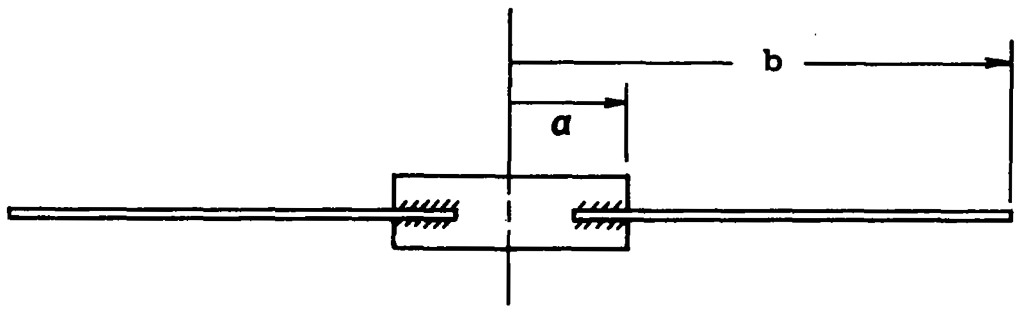
Figure 3. Coordinate System in which Displacements are Measured Radially, Tangentially, and Axially.



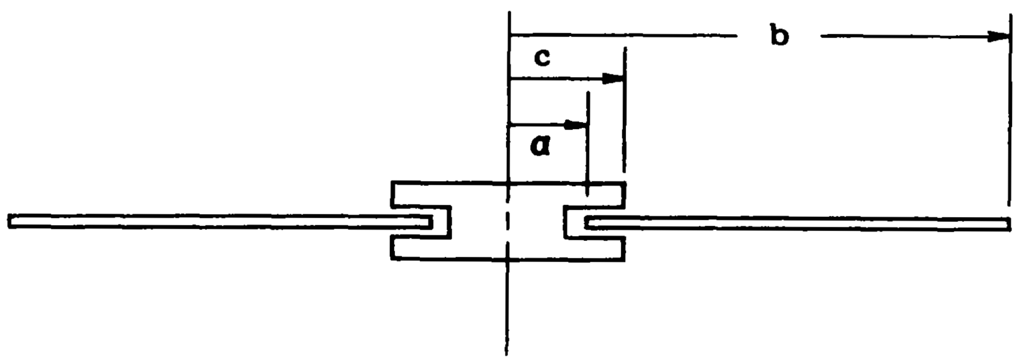
$$F_I = m\omega^2 (r + \Delta r)\ t r d r d\theta$$

$$\Delta r = u + w\frac{r}{R}$$

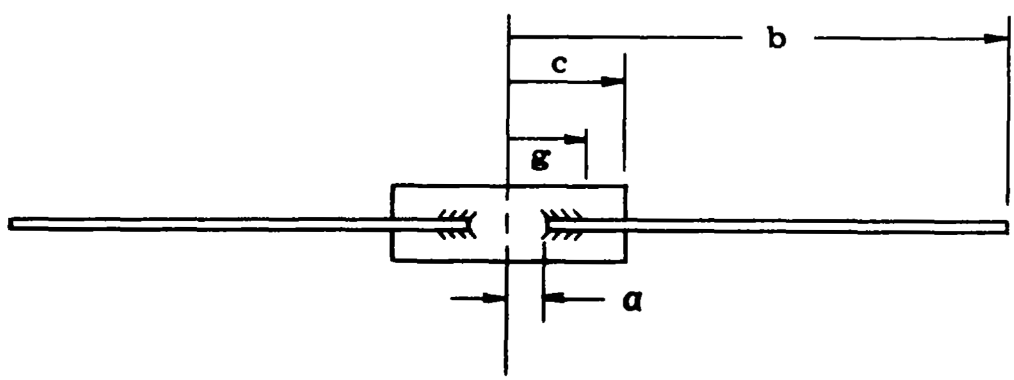
Figure 4. Geometry of the Inertia Load.



(a) HUB FULLY CLAMPED AT RADIUS $r = a$



(b) ANNULAR DISK WITH INNER RADIUS $r = a$ LOOSELY CLAMPED BY HUB OR RADIUS $r = c$



(c) ANNULAR DISK WITH INNER RADIUS $r = a$ CLAMPED BY HUB OF $r = c$ SO THAT FULL CLAMPING EXISTS UP TO $r = g$ AND PARTIAL CLAMPING EXISTS FOR $g \leq r \leq c$

Figure 5. Hub Configurations

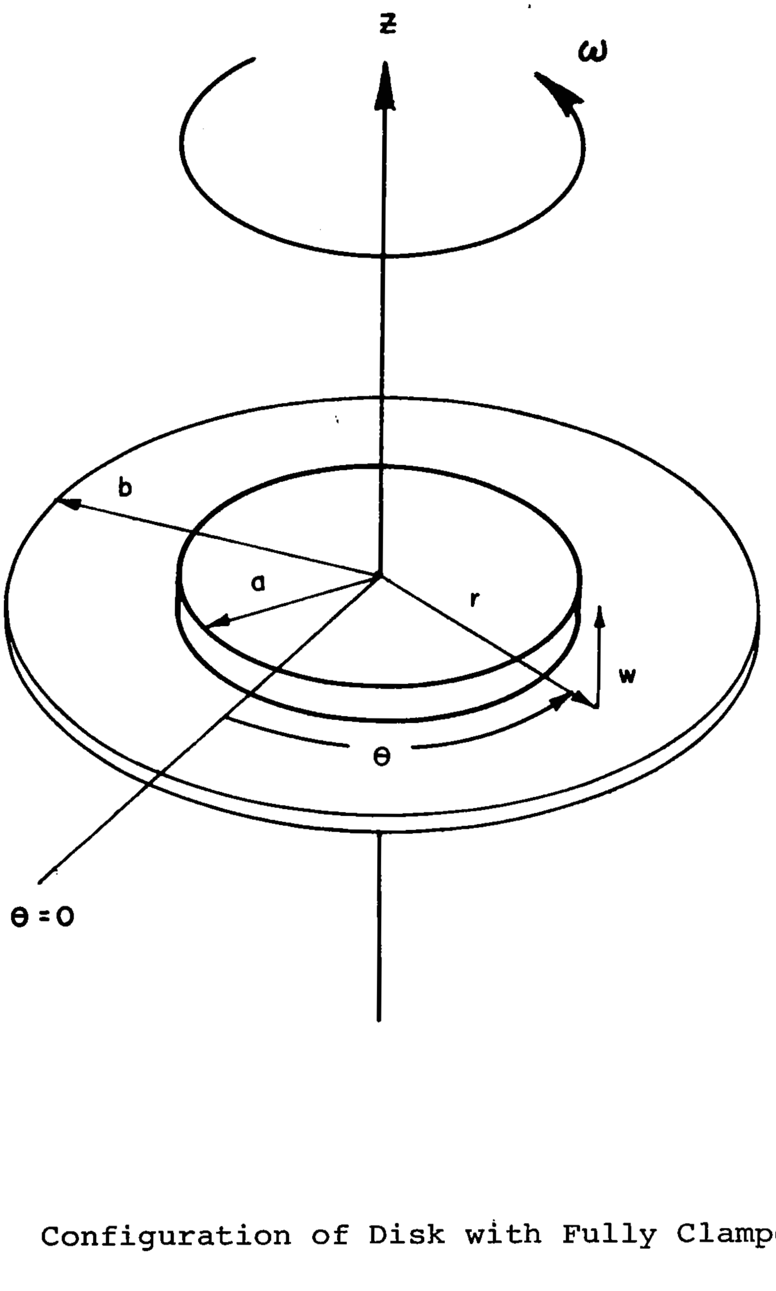


Figure 6. Configuration of Disk with Fully Clamped Hub.

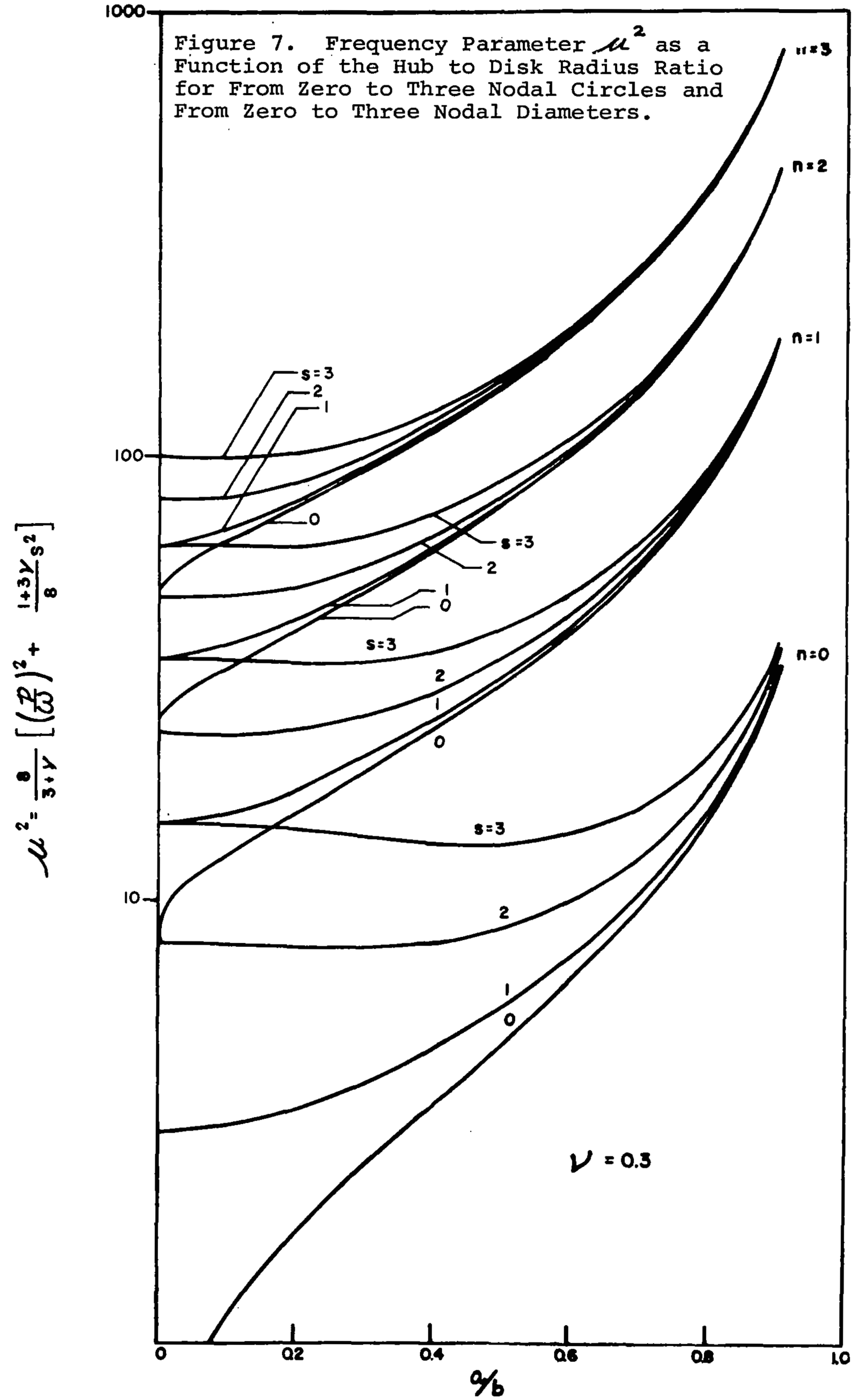


Figure 7. Frequency Parameter μ^2 as a Function of the Hub to Disk Radius Ratio for From Zero to Three Nodal Circles and From Zero to Three Nodal Diameters.

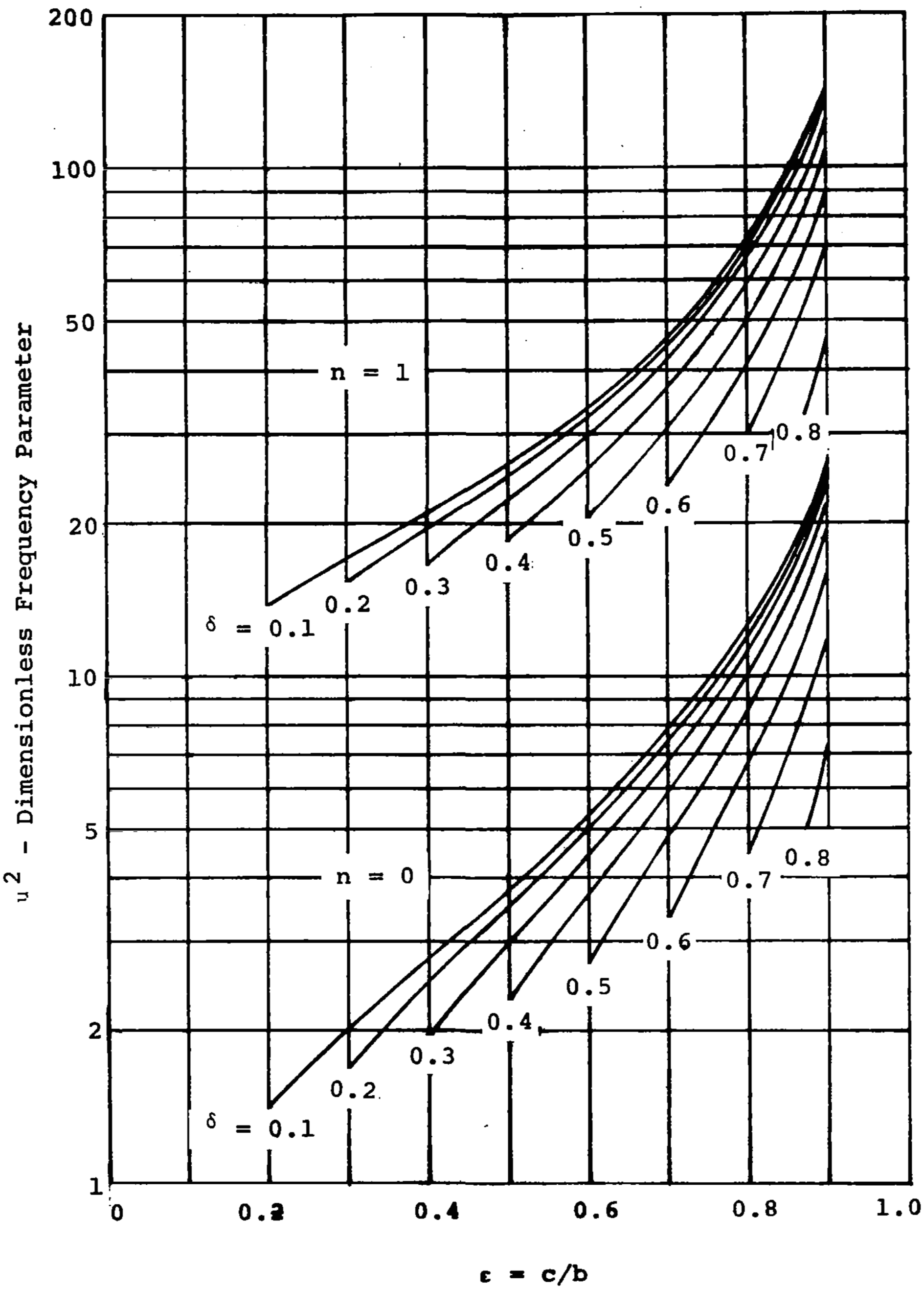


Figure 8. Variation of the Frequency Parameter, μ^2, as a Function of the Hub to Disk Radius Ratio, ε, for Various Values of the Annulus Radius Ratio, δ, for the Symmetric Vibration Case, s=0, and for the Cases of Zero and One Nodal Circles, n=0 and 1.

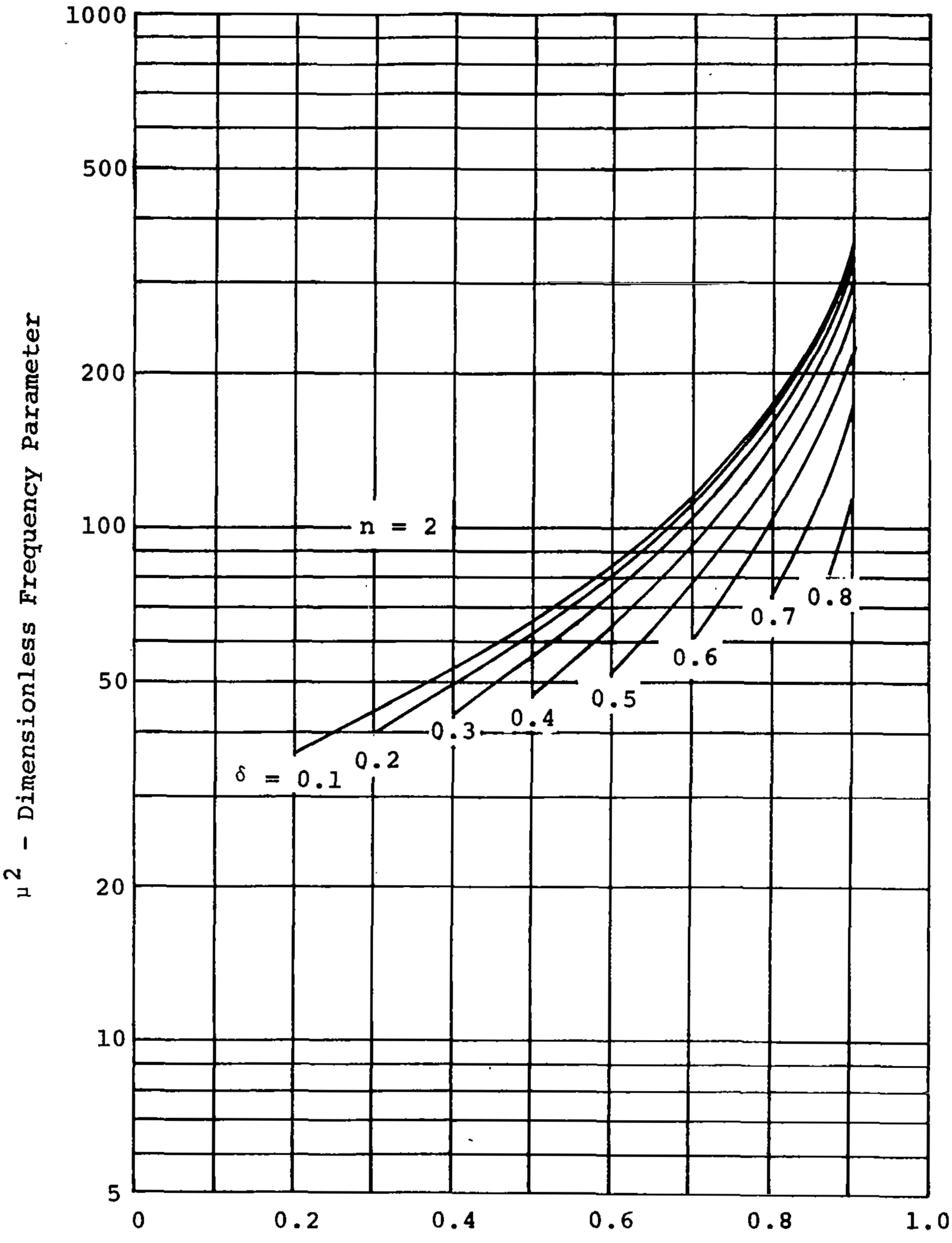


Figure 9. Variation of the Frequency Parameter, μ^2, as a Function of the Hub to Disk Radius Ratio, ε, for Various Values of the Annulus Radius Ratio, δ, for the Symmetric Vibration Case, s=0, and for the Case of Two Nodal Circles, n=2.

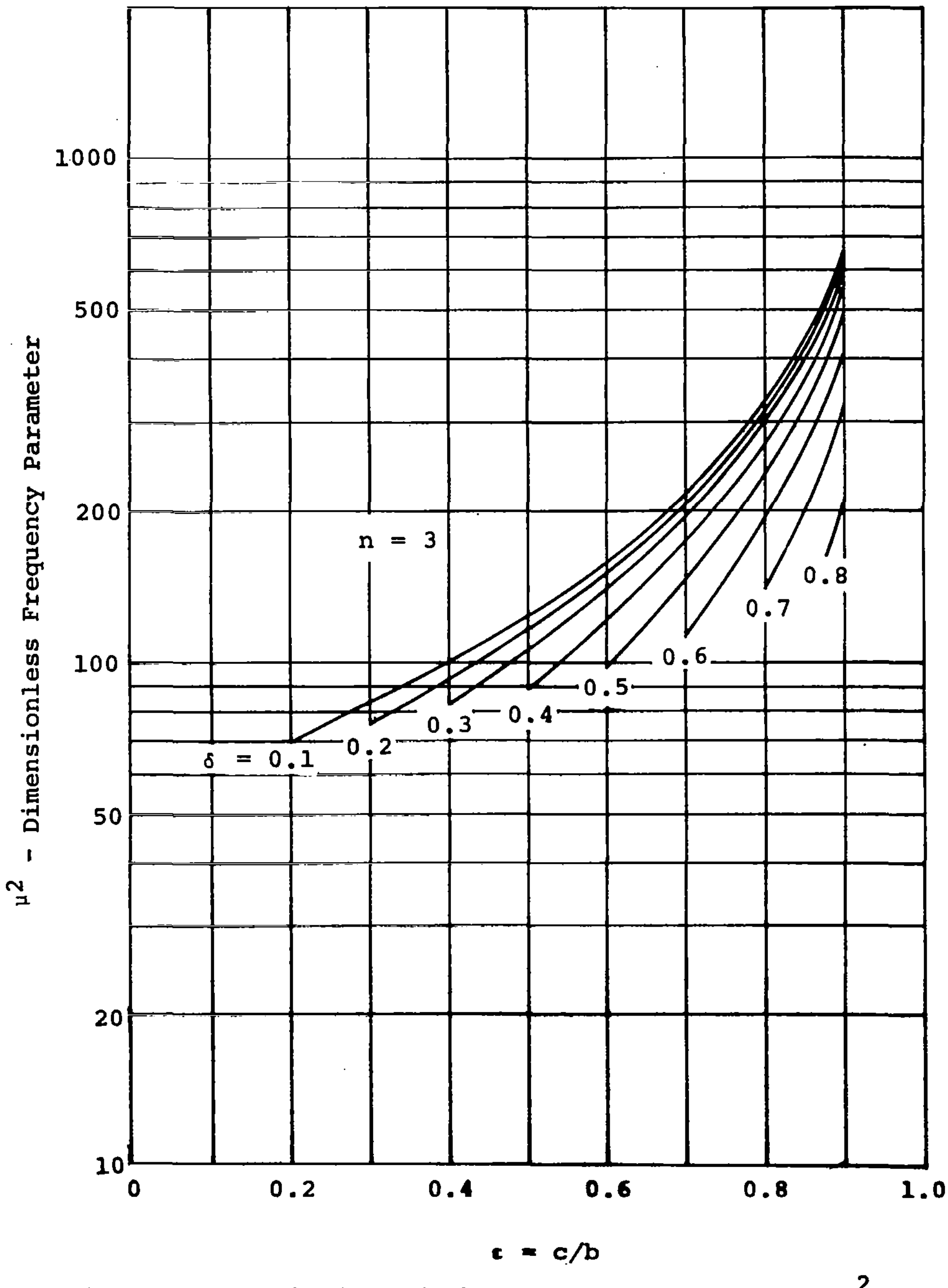


Figure 10. Variation of the Frequency Parameter, μ^2, as a Function of the Hub to Disk Radius Ratio, ε, for Various Values of the Annulus Radius Ratio, δ, for the Symmetric Vibration Case, s=0 and for the Case of Three Nodal Circles, n=3.

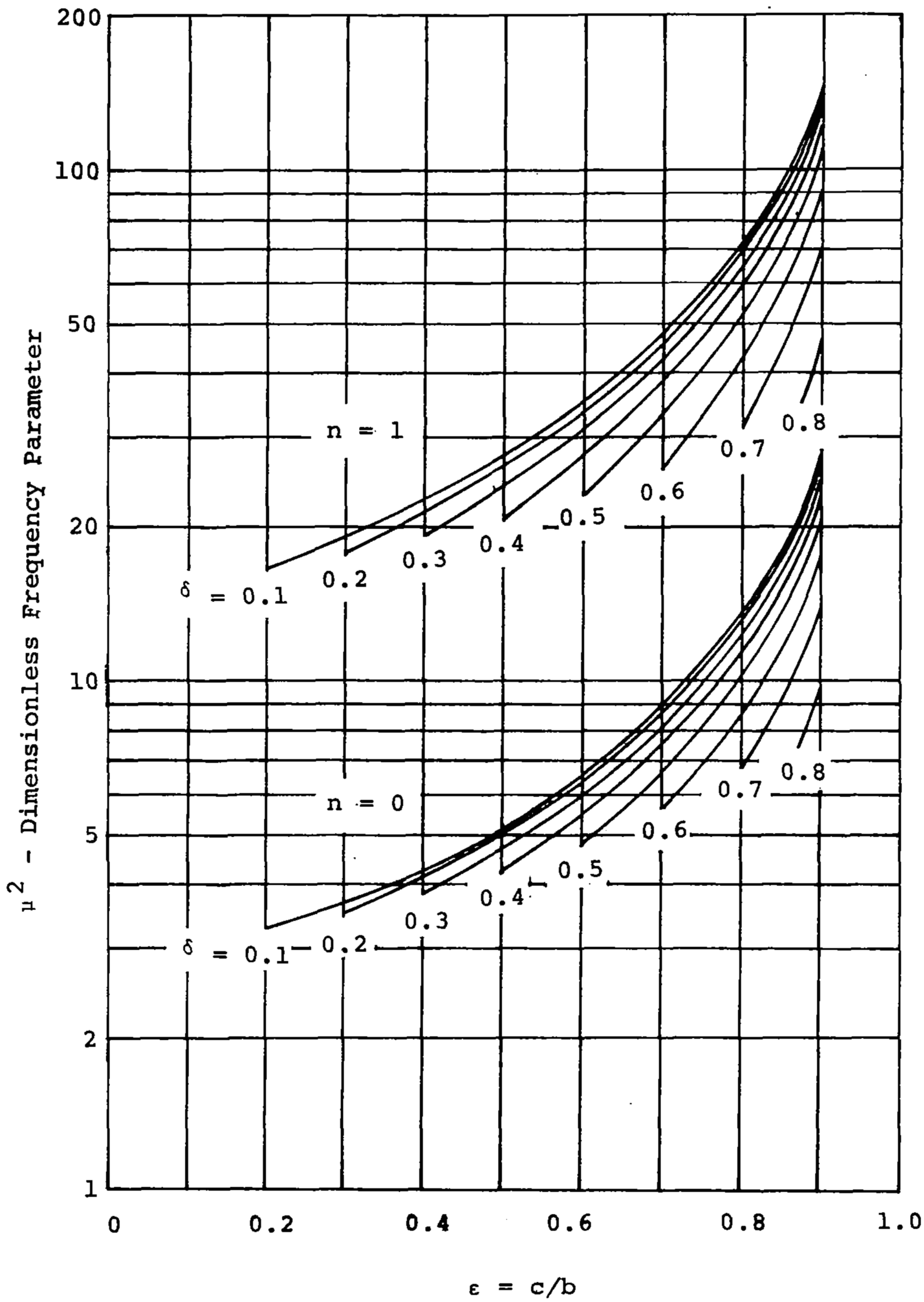


Figure 11. Variation of the Frequency Parameter, μ^2 as a Function of the Hub of Disk Radius Ratio, ε, for Various Values of the Annulus Radius Ratio, δ, for the Case of One Nodal Diameter, s=1, and for the cases of Zero and One Nodal Circles, n=0 and 1.

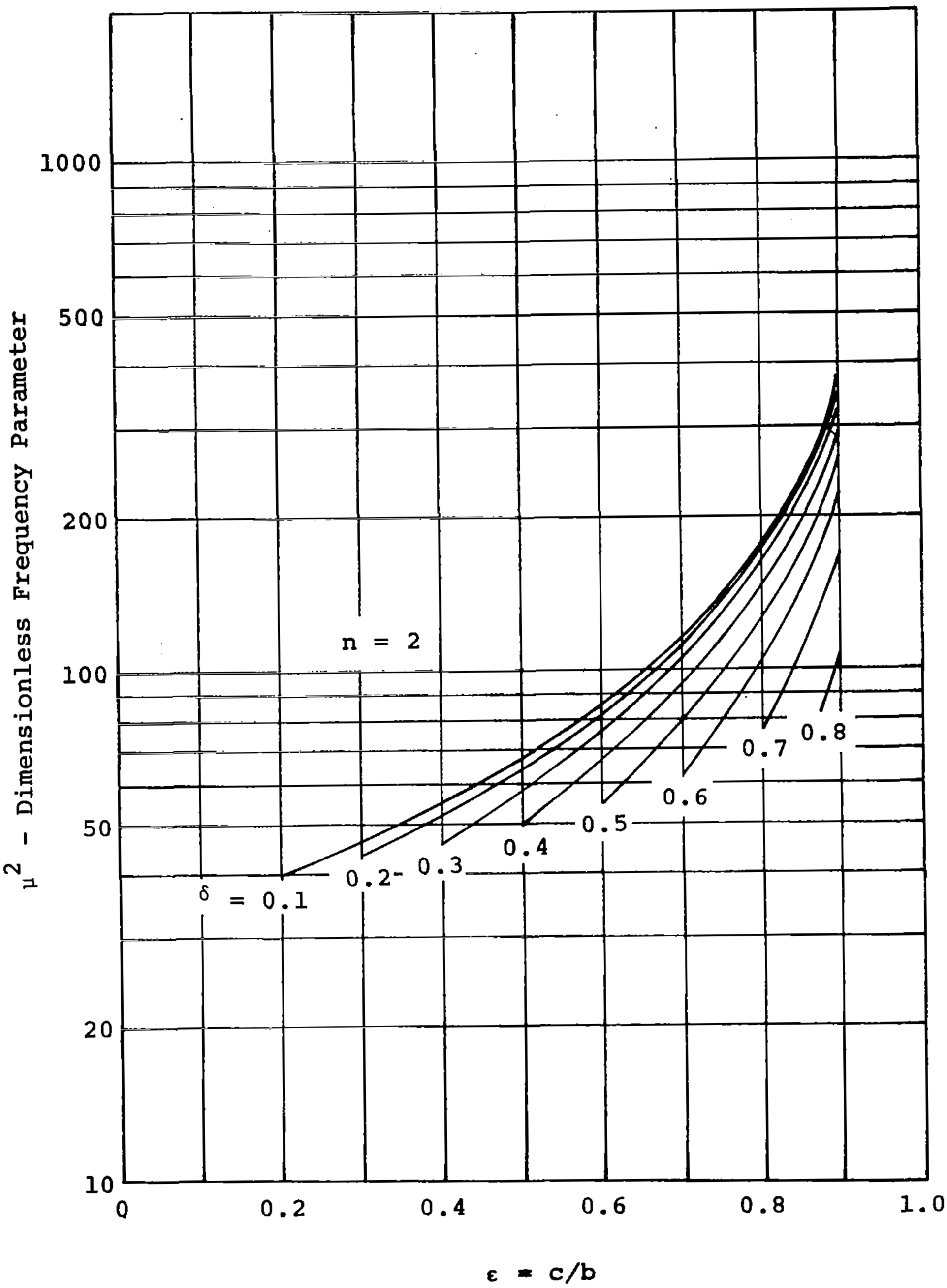


Figure 12. Variation of the Frequency Parameter, μ^2, as a Function of the Hub to Disk Radius Ratio, ϵ, for Various Values of the Annulus Radius Ratio, δ, for the Case of One Nodal Diameter, s=1, and for the Case of Two Nodal Circles, n=2.

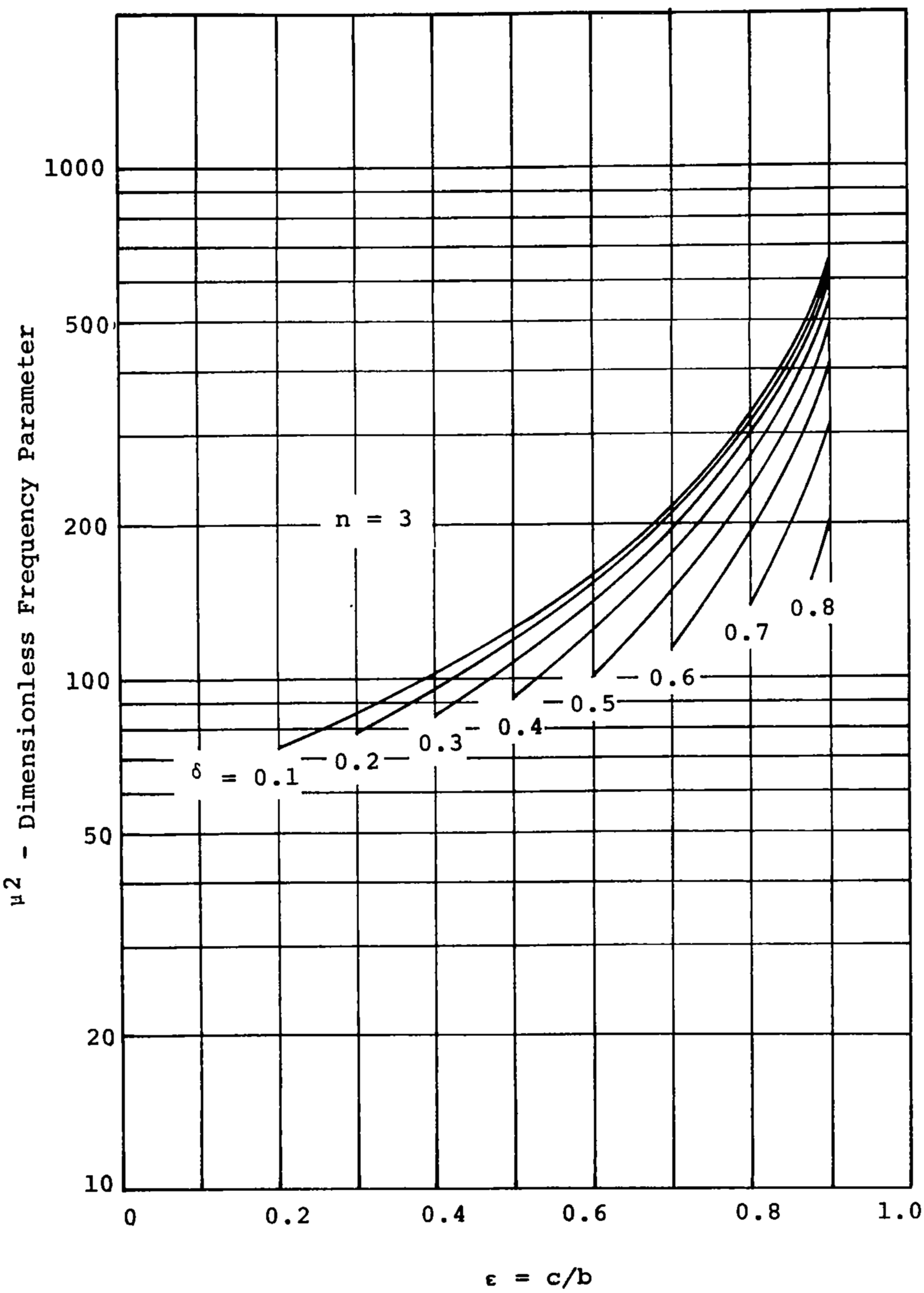


Figure 13. Variation of the Frequency Parameter, μ^2, as a Function of the Hub to Disk Radius Ratio, ε, for Various Values of the Annulus Radius Ratio, δ, for the Case of One Nodal Diameter, s=1, and for the Case of Three Nodal Circles, n=3.

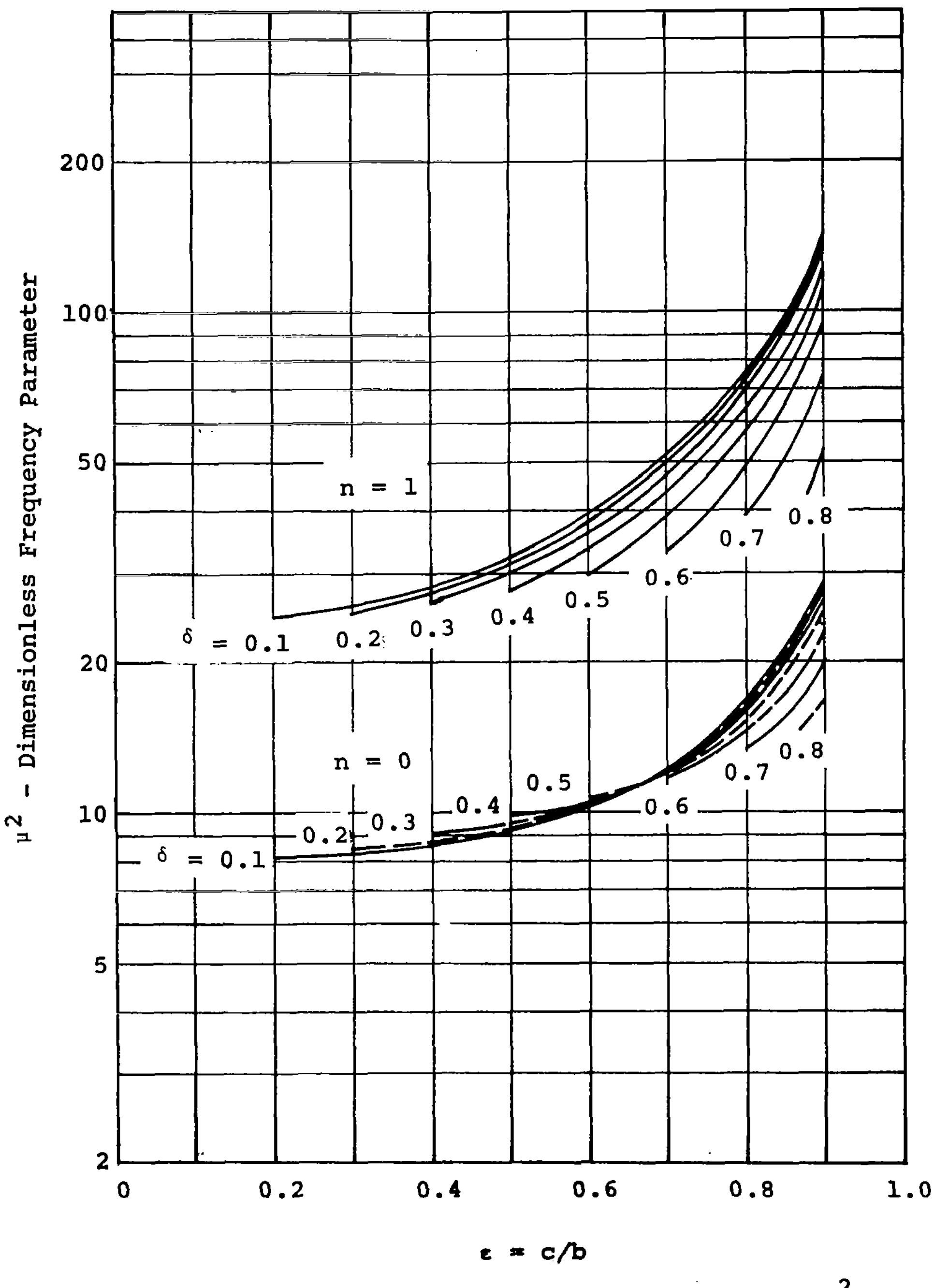


Figure 14. Variation of the Frequency Parameter, μ^2, as a Function of the Hub to Disk Radius Ratio, ε, for Various Values of the Annulus Radius Ratio, δ, for the Case of Two Nodal Diameters, s=2, and for the Cases of Zero of One Nodal Circles, n=0 and 1.

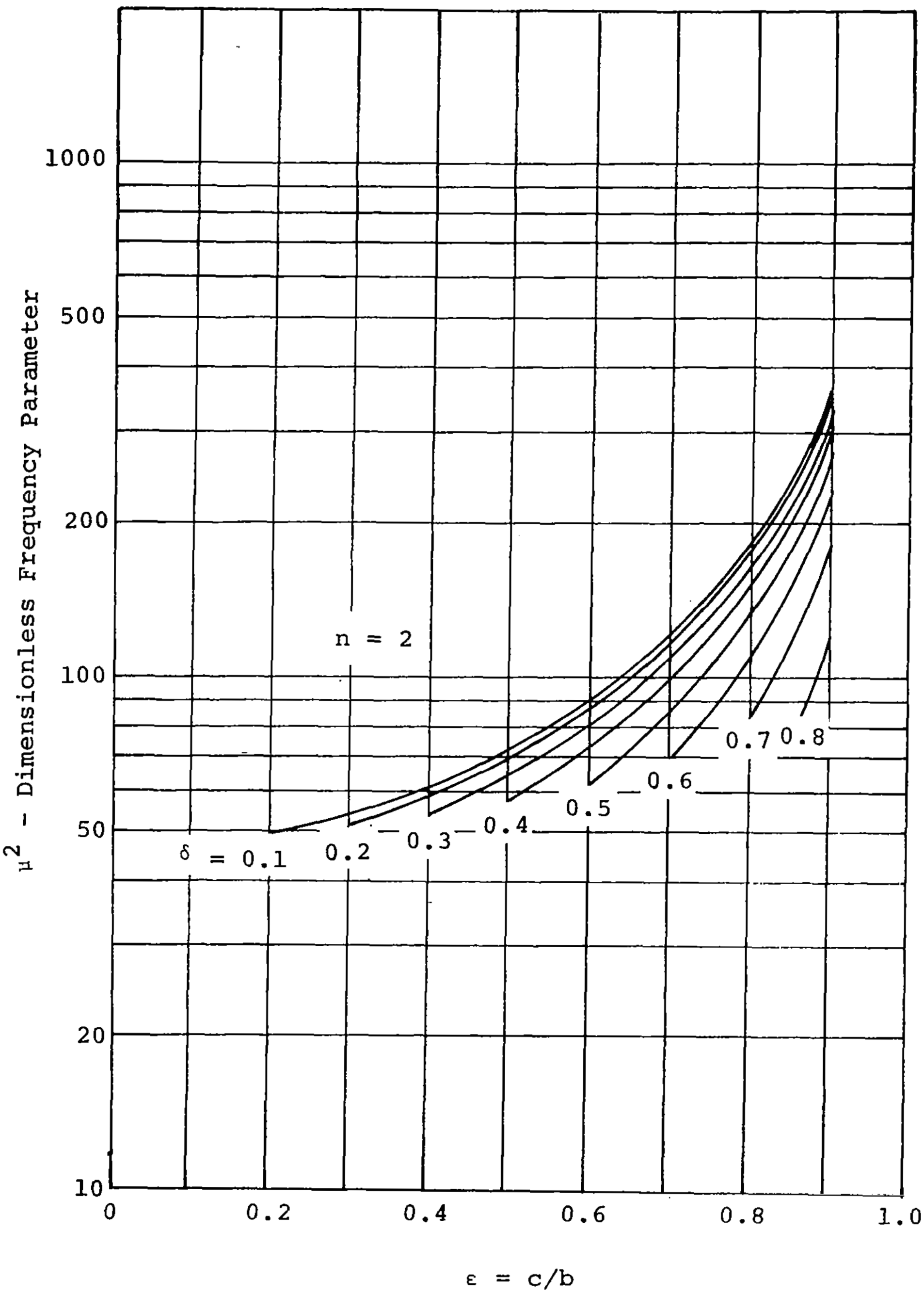


Figure 15. Variation of the Frequency Parameter, μ^2, as a Function of the Hub to Disk Radius Ratio, ϵ, for Various Values of the Annulus Radius Ratio, δ, for the Case of Two Nodal Diameters, s=2, and for the Case of Two Nodal Circles, n=2.

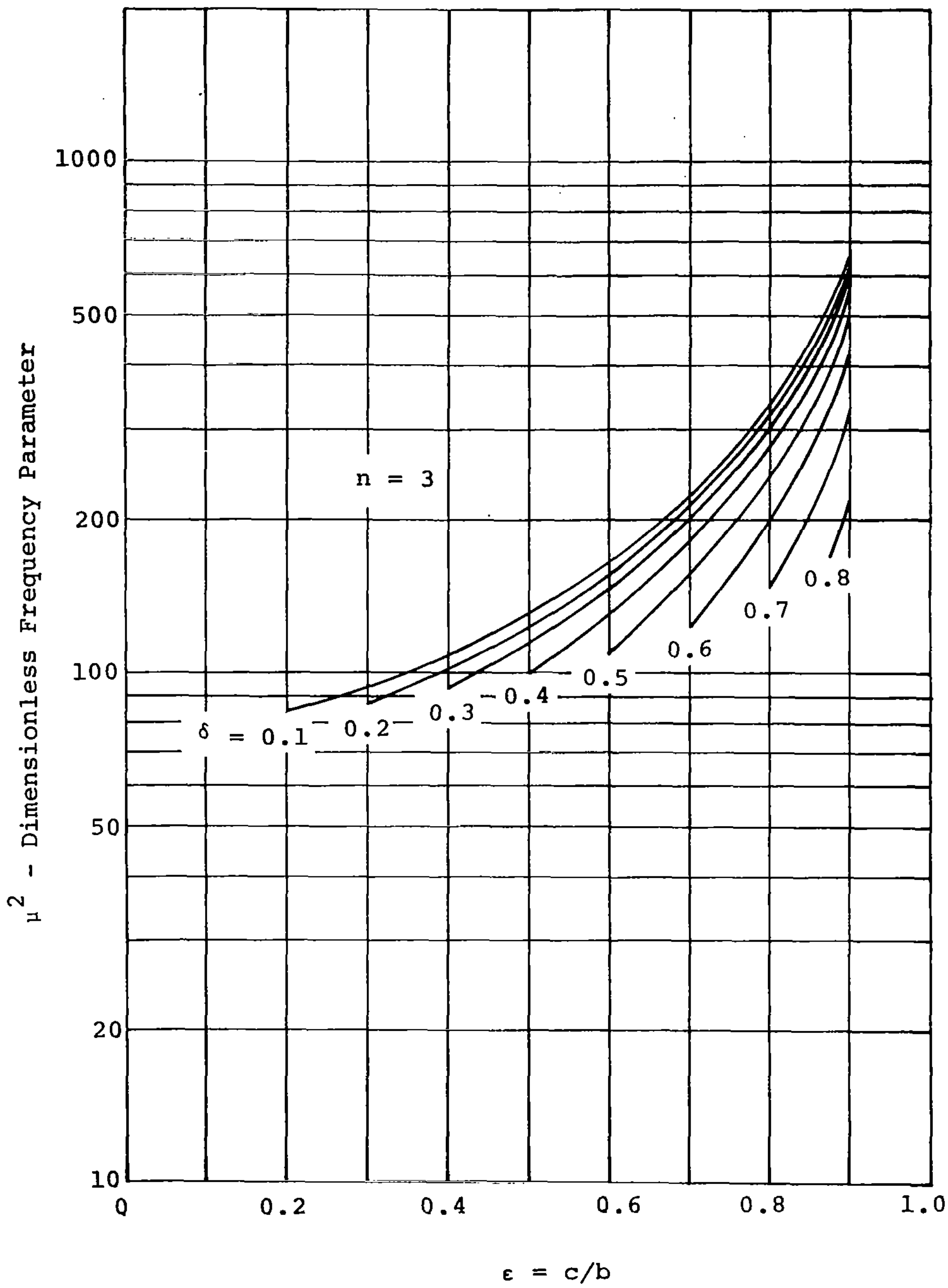


Figure 16. Variation of the Frequency Parameter, μ^2, as a Function of the Hub to Disk Radius Ratio, ϵ, for Various Values of the Annulus Radius Ratio, δ, for the Case of Two Nodal Diameters, s=2, and for the Case of Three Nodal Circles, n=3.

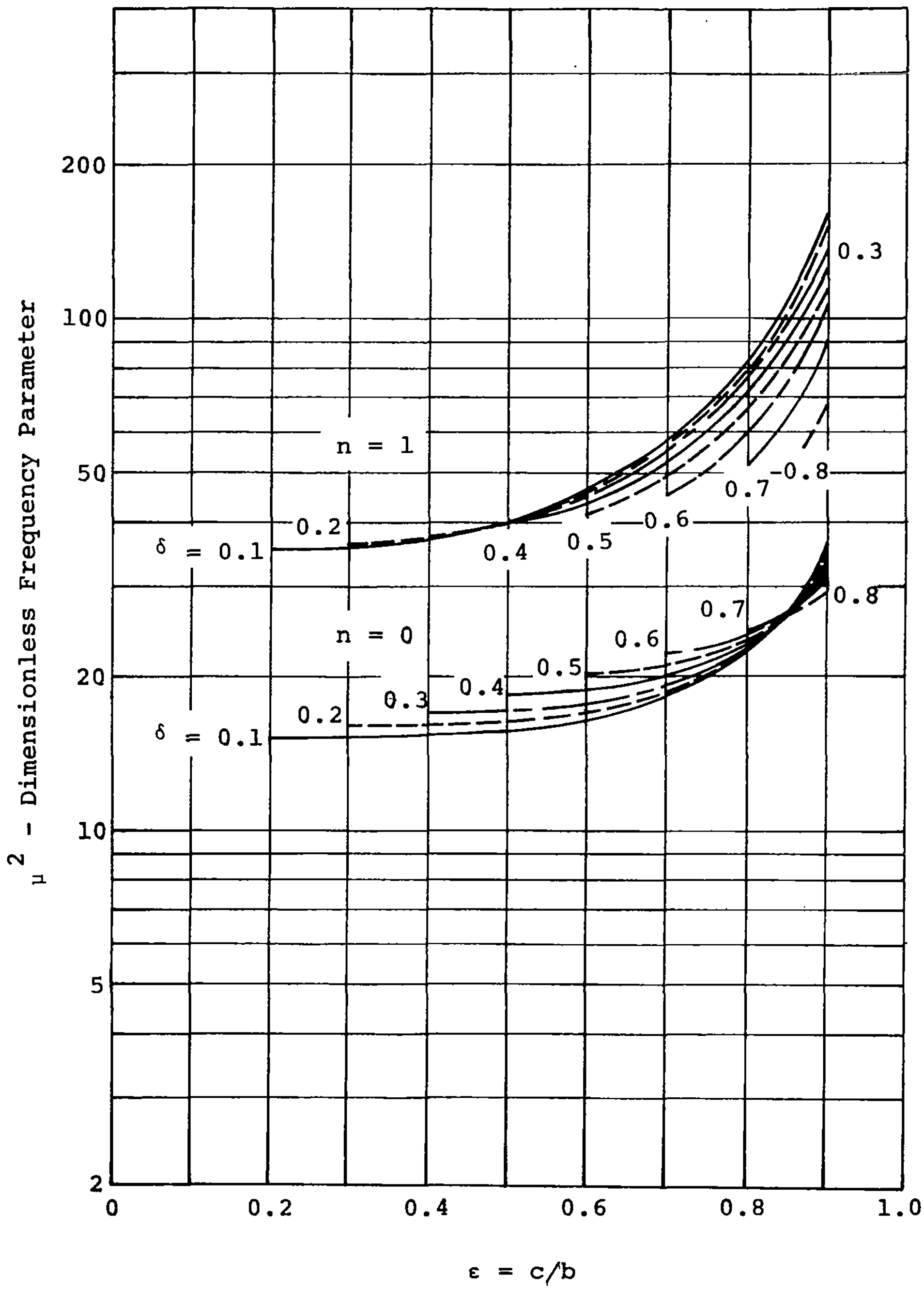


Figure 17. Variation of the Frequency Parameter, μ^2, as a Function of the Hub to Disk Radius Ratio, ε, for Various Values of the Annulus Radius Ratio, δ, for the Case of Three Nodal Diameters, s=3 and for the Cases of Zero and One Nodal Circles, n=0 and 1.

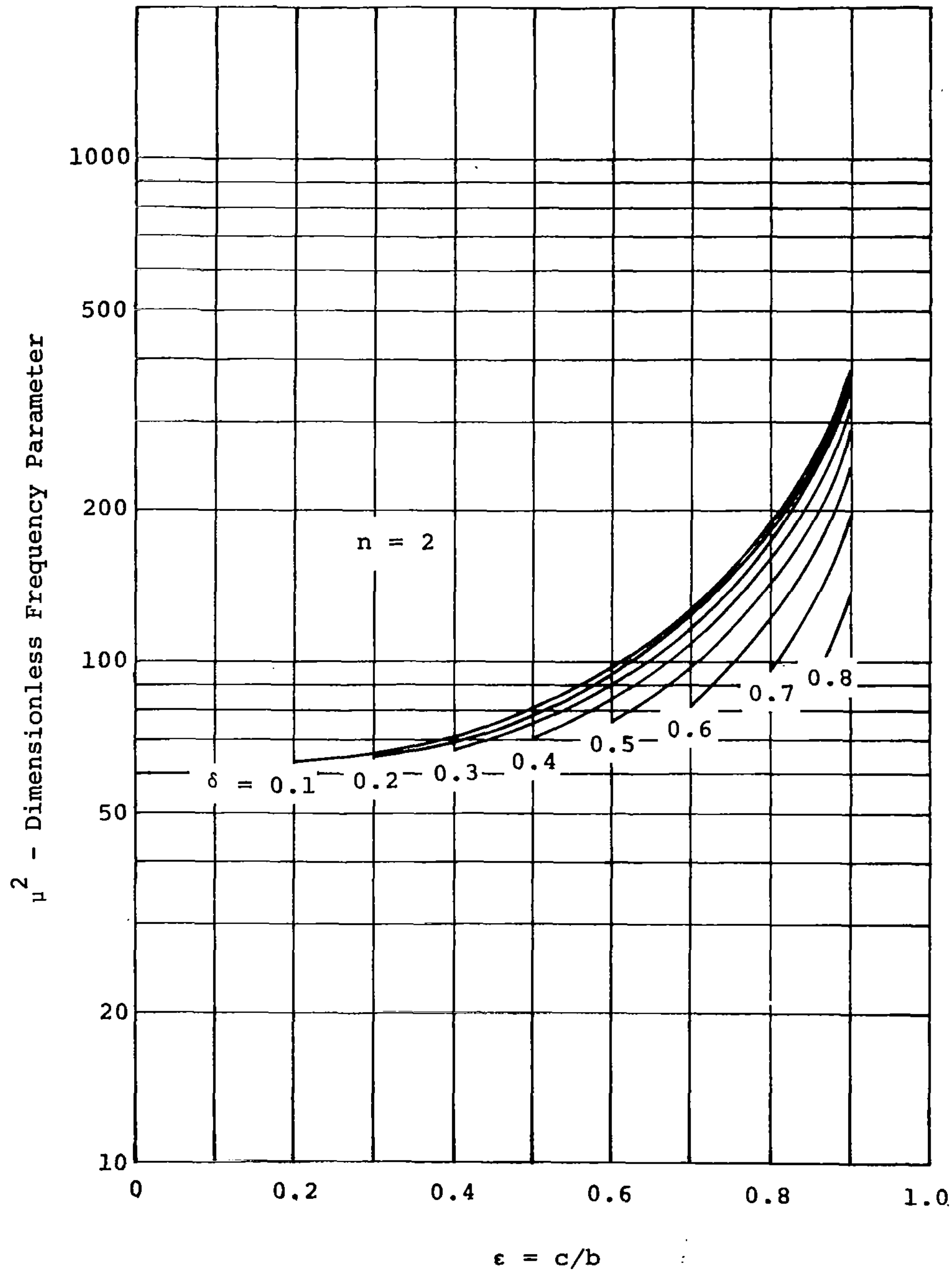


Figure 18. Variation of the Frequency Parameter, μ^2, as a Function of the Hub to Disk Radius Ratio, ε, for Various Values of the Annulus Radius Ratio, δ, for the Case of Three Nodal Diameters, s=3, and for the Case of Two Nodal Circles, n=2.

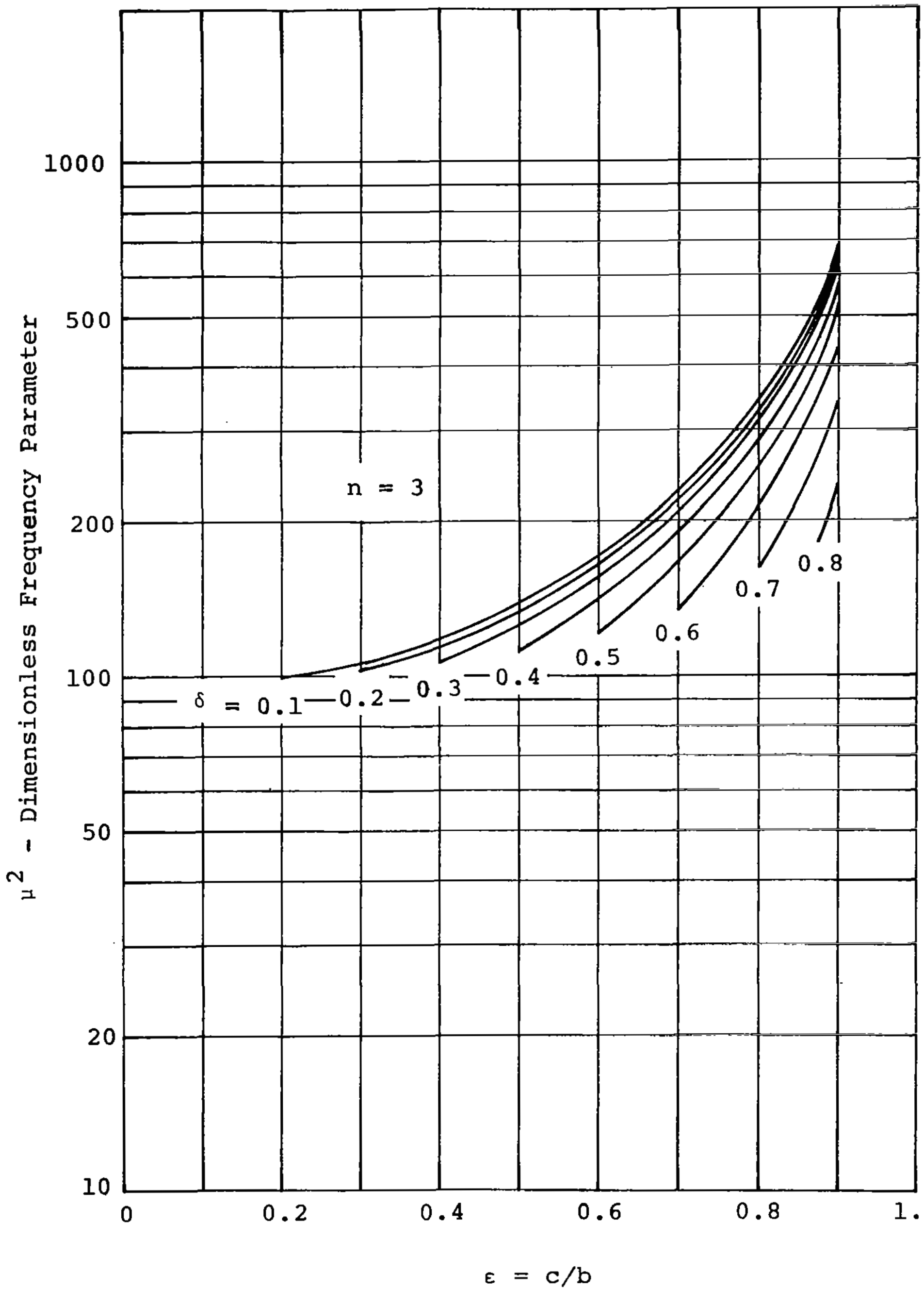


Figure 19. Variation of the Frequency Parameter, μ^2, as a Function of the Hub to Disk Radius Ratio, ε, for Various Values of the Annulus Radius Ratio, δ, for the Case of Three Nodal Diameters, s=3, and for the Case of Three Nodal Circles, n=3.

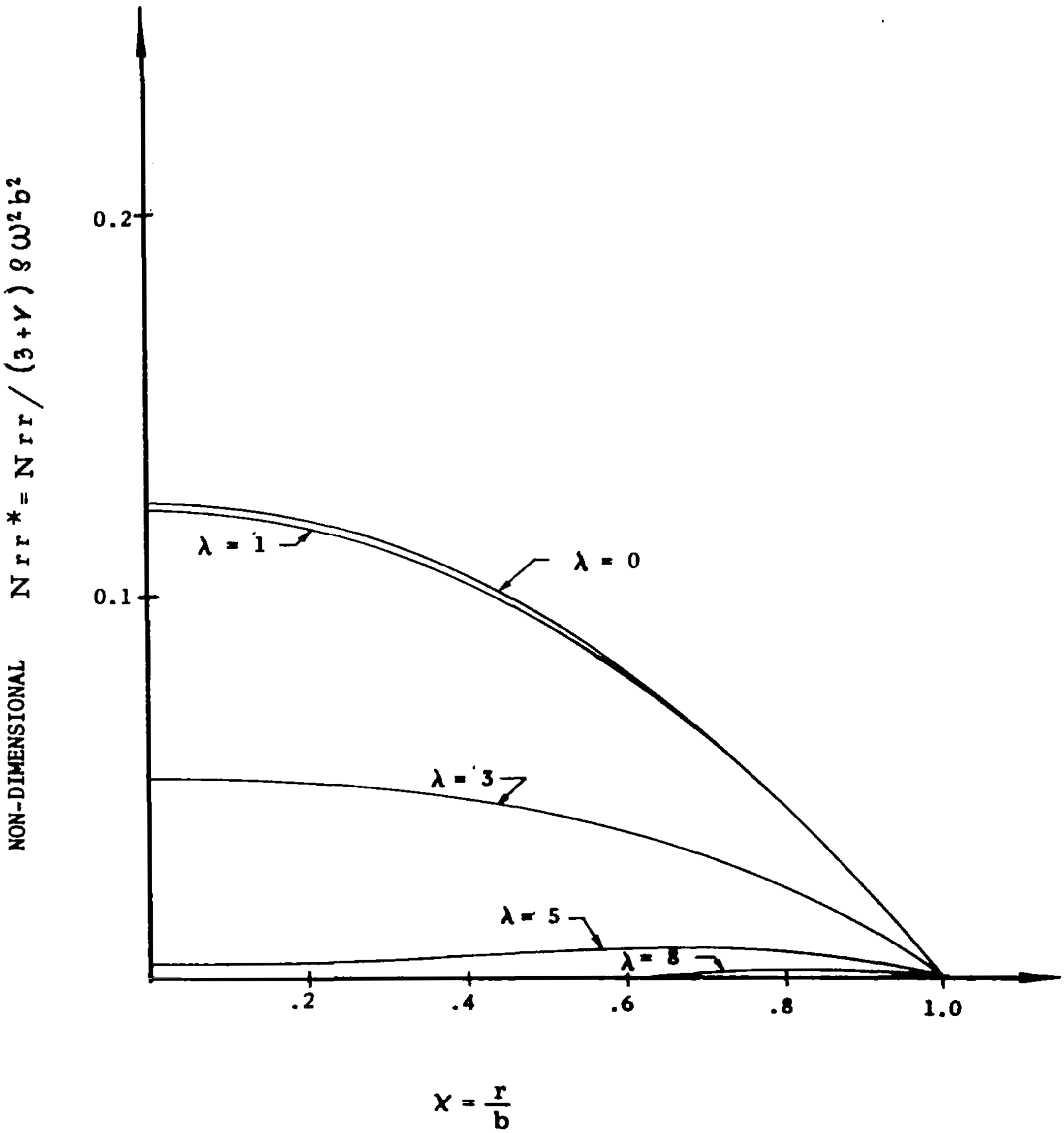


Figure 20. Radial Direct Stress Resultant for Freely Spinning Shell. Linear Theory with ν =0.20.

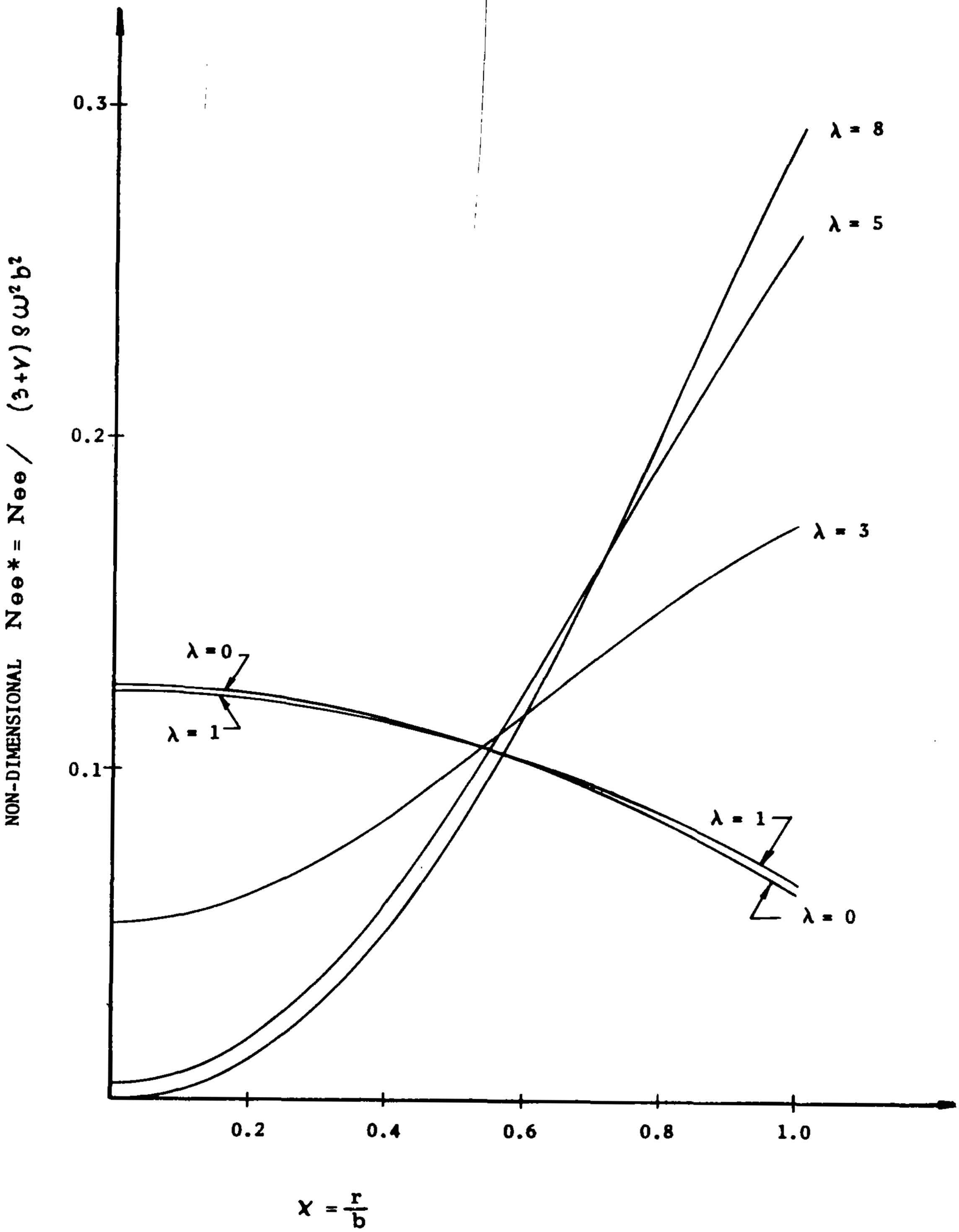


Figure 21. Tangential Direct Stress Resultant for Freely Spinning Shell. Linear Theory with ν =0.20.

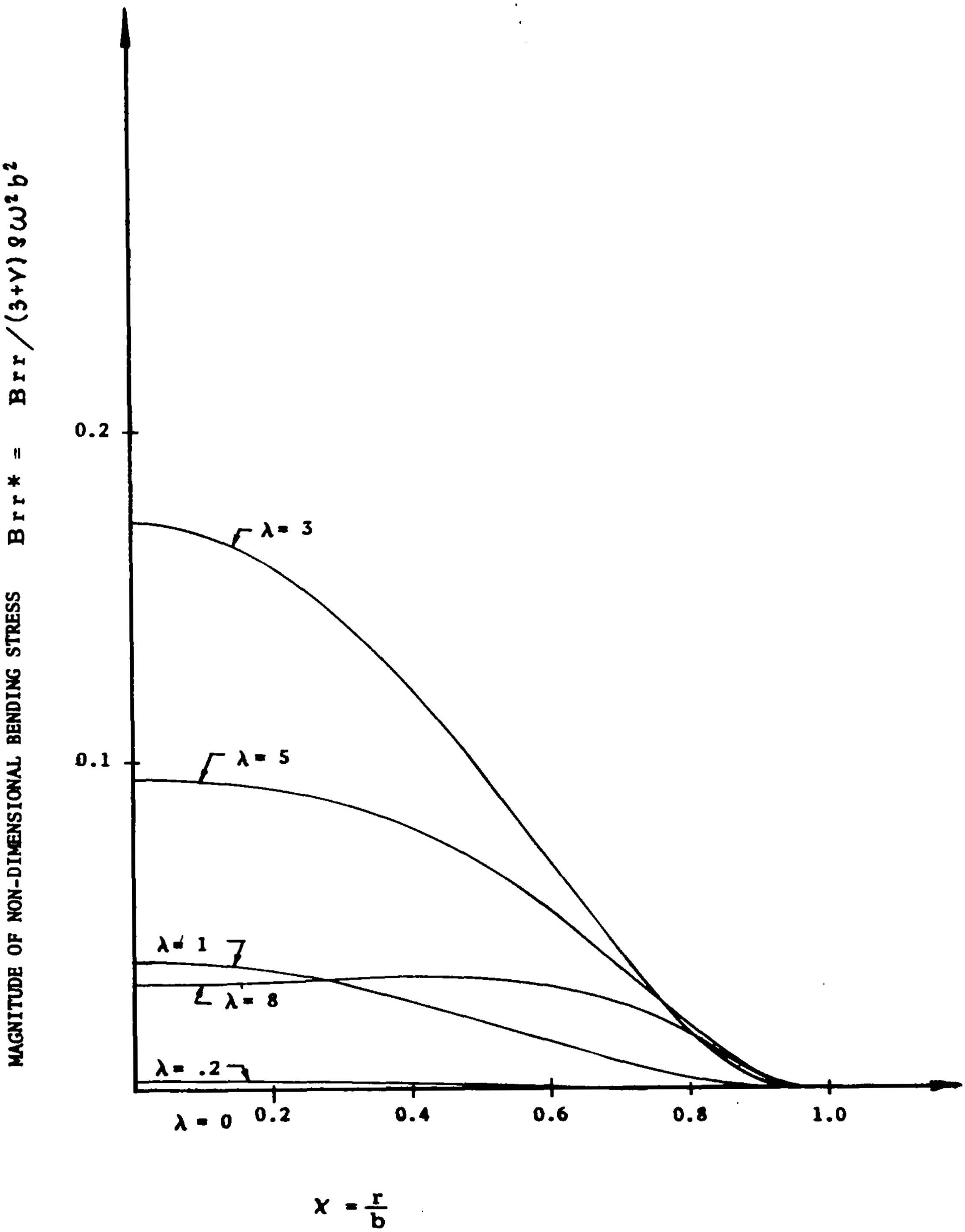


Figure 22. Radial Bending Stress Resultant for Freely Spinning Shell. Linear Theory with ν =0.20.

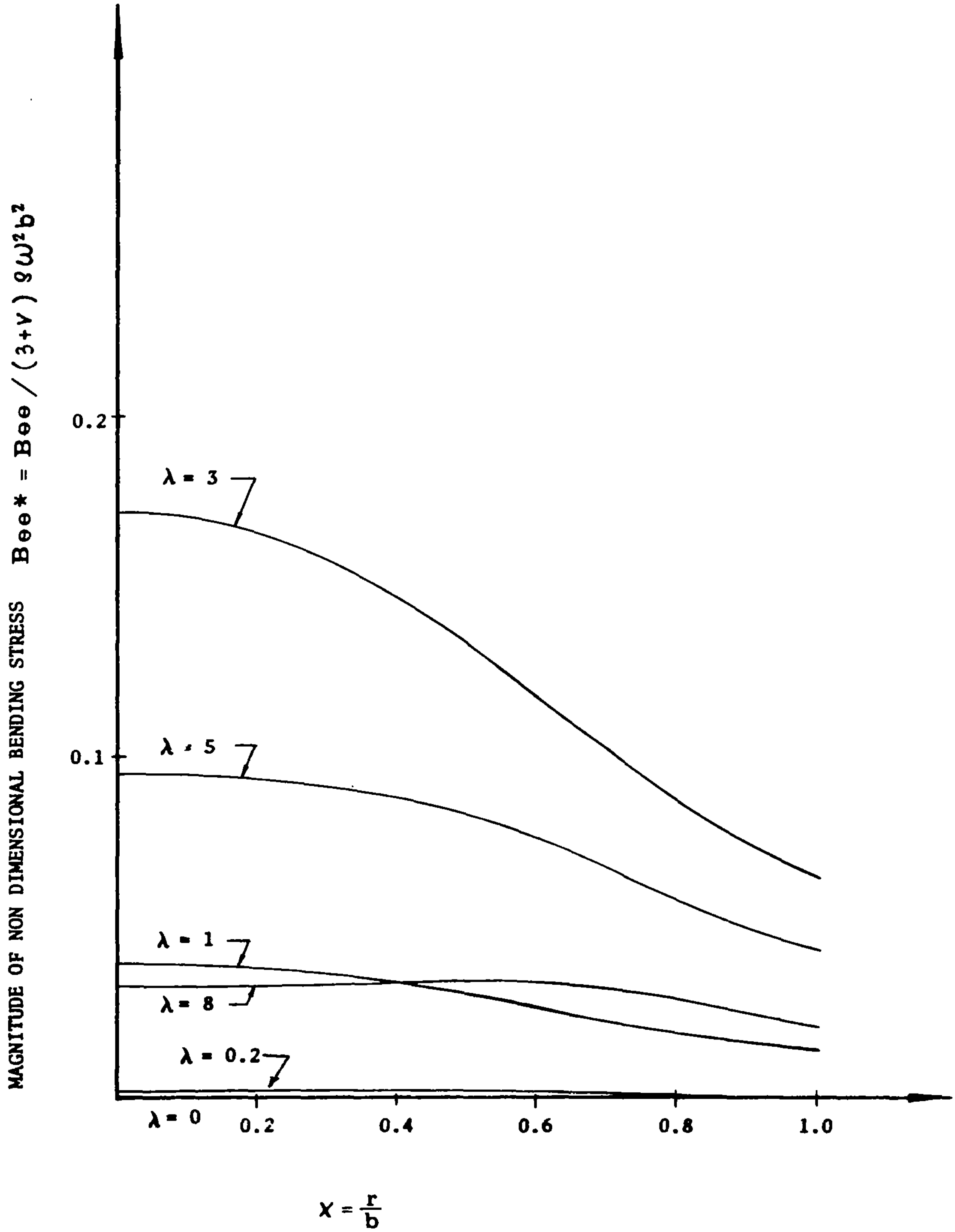


Figure 23. Tangential Bending Stress Resultant for Freely Spinning Shell. Linear Theory with ν =0.20.

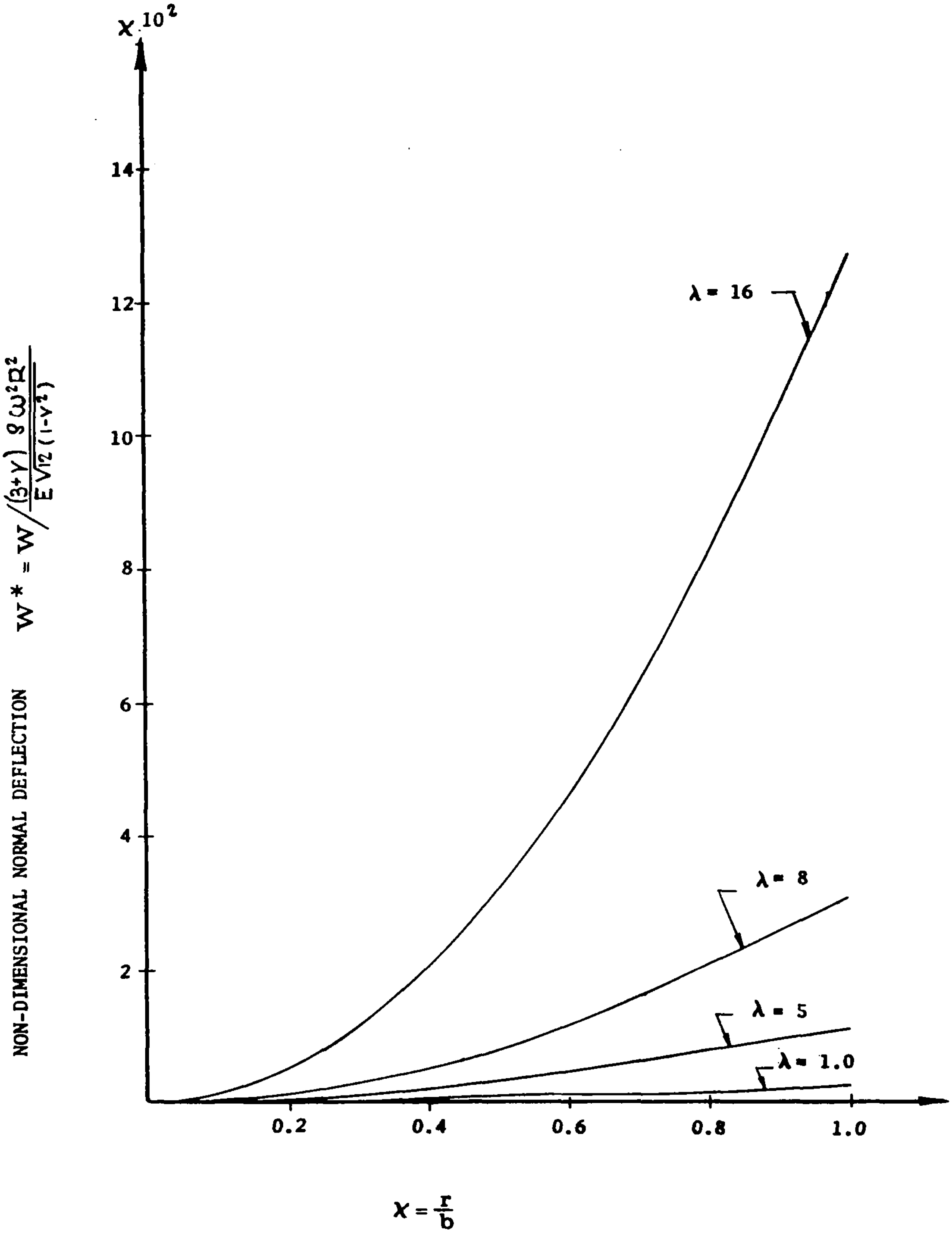


Figure 24. Normal Deflection for Freely Spinning Shell. Linear Theory with ν =0.20.

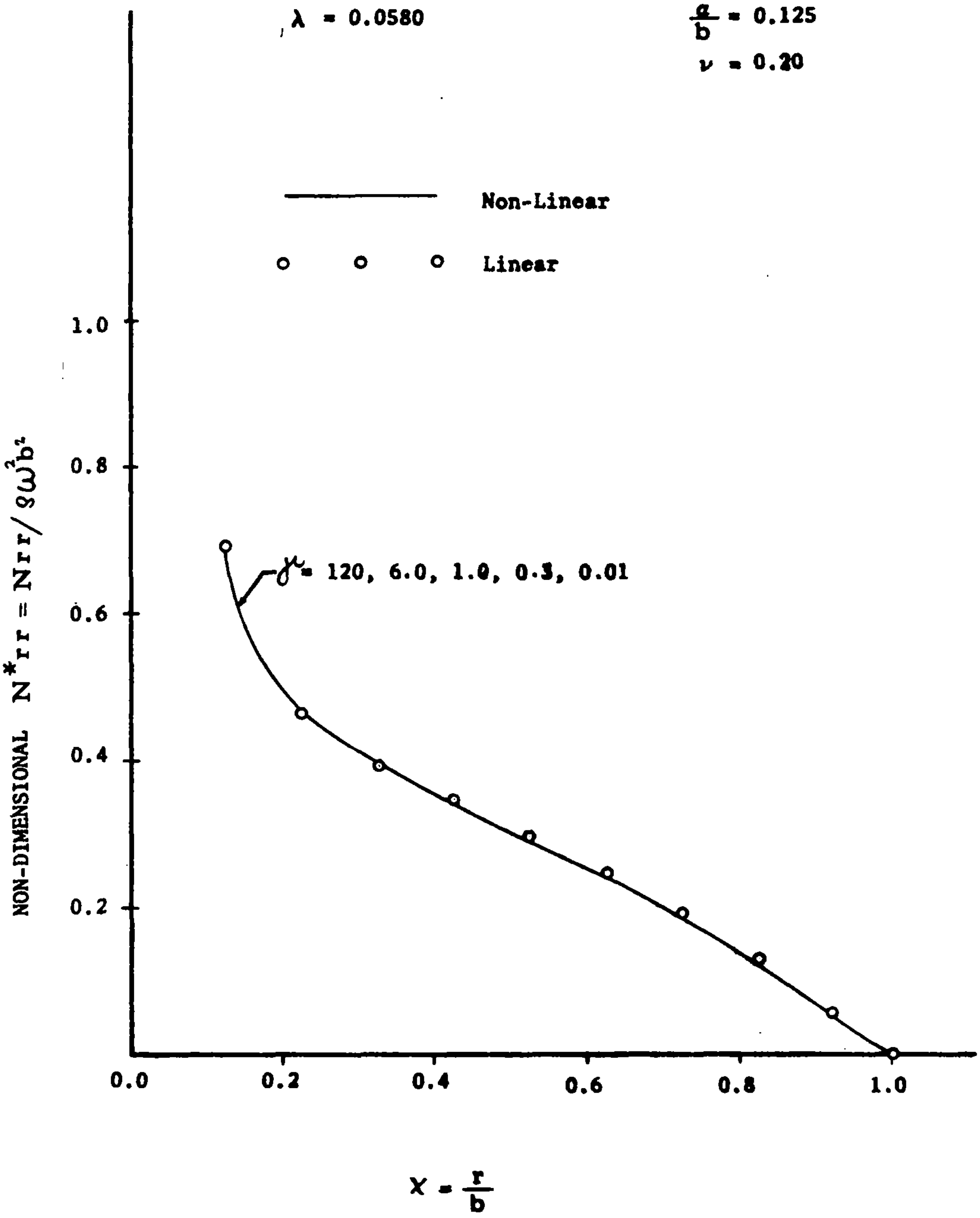


Figure 25. Radial Direct Stress Resultant for Spinning Shell with Fully Clamped Hub. Linear and Nonlinear Theory with λ =0.058 and ν =0.20.

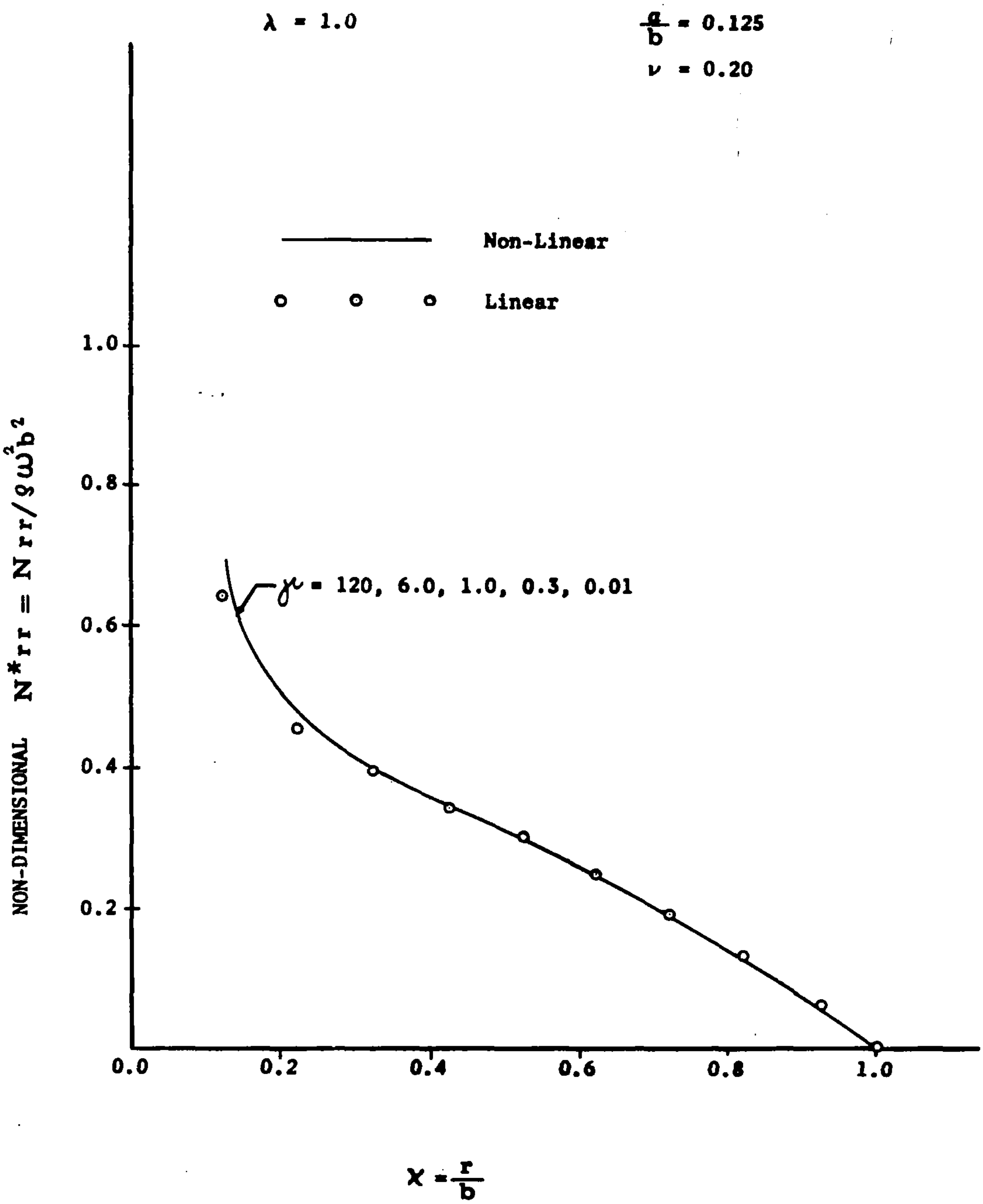


Figure 26. Radial Direct Stress Resultant for Spinning Shell with Fully Clamped Hub. Linear and Nonlinear Theory with λ =1.0 and ν =0.20.

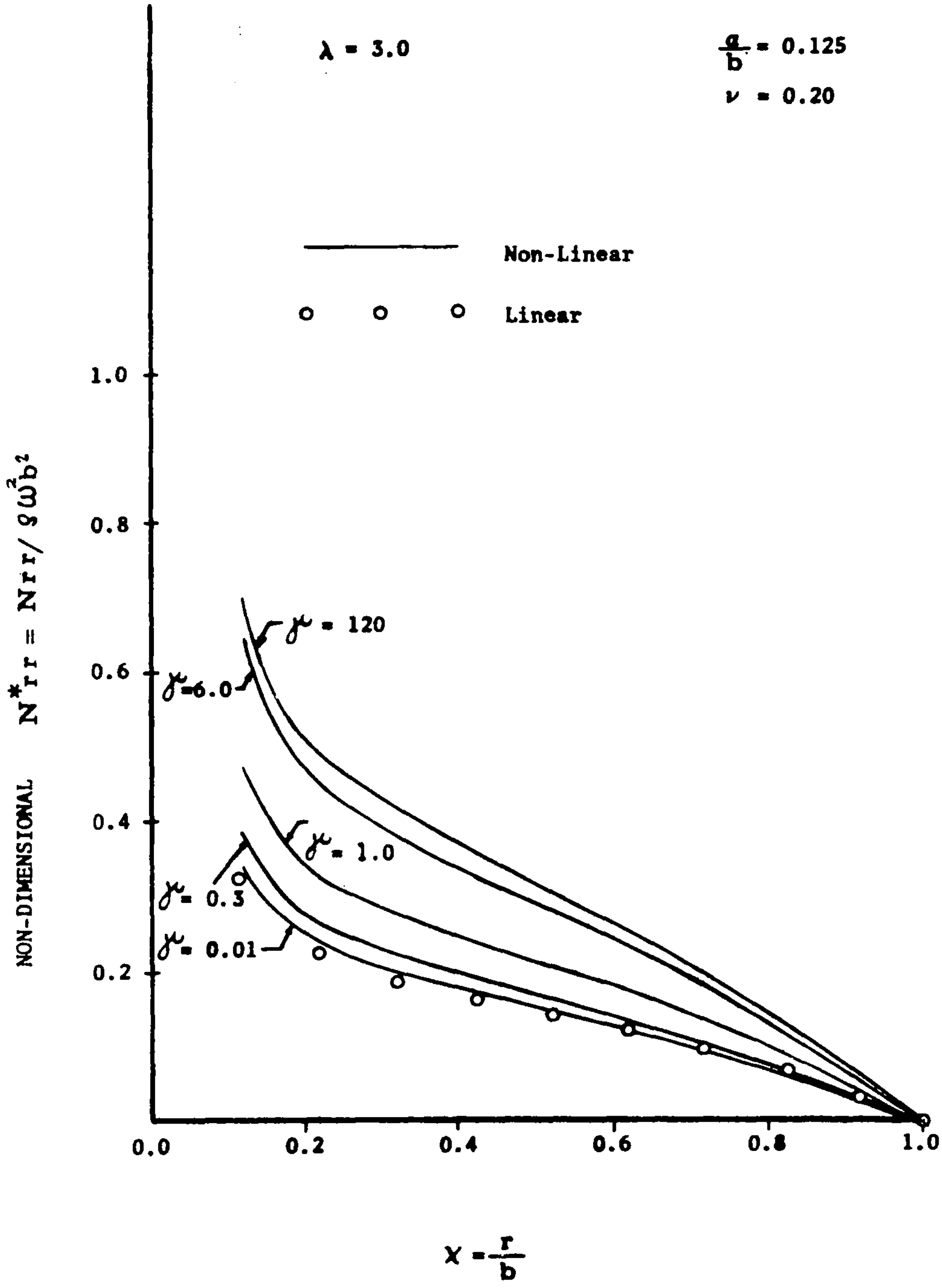


Figure 27. Radial Direct Stress Resultant for Spinning Shell with Fully Clamped Hub. Linear and Nonlinear Theory with λ =3.0 and ν =0.20.

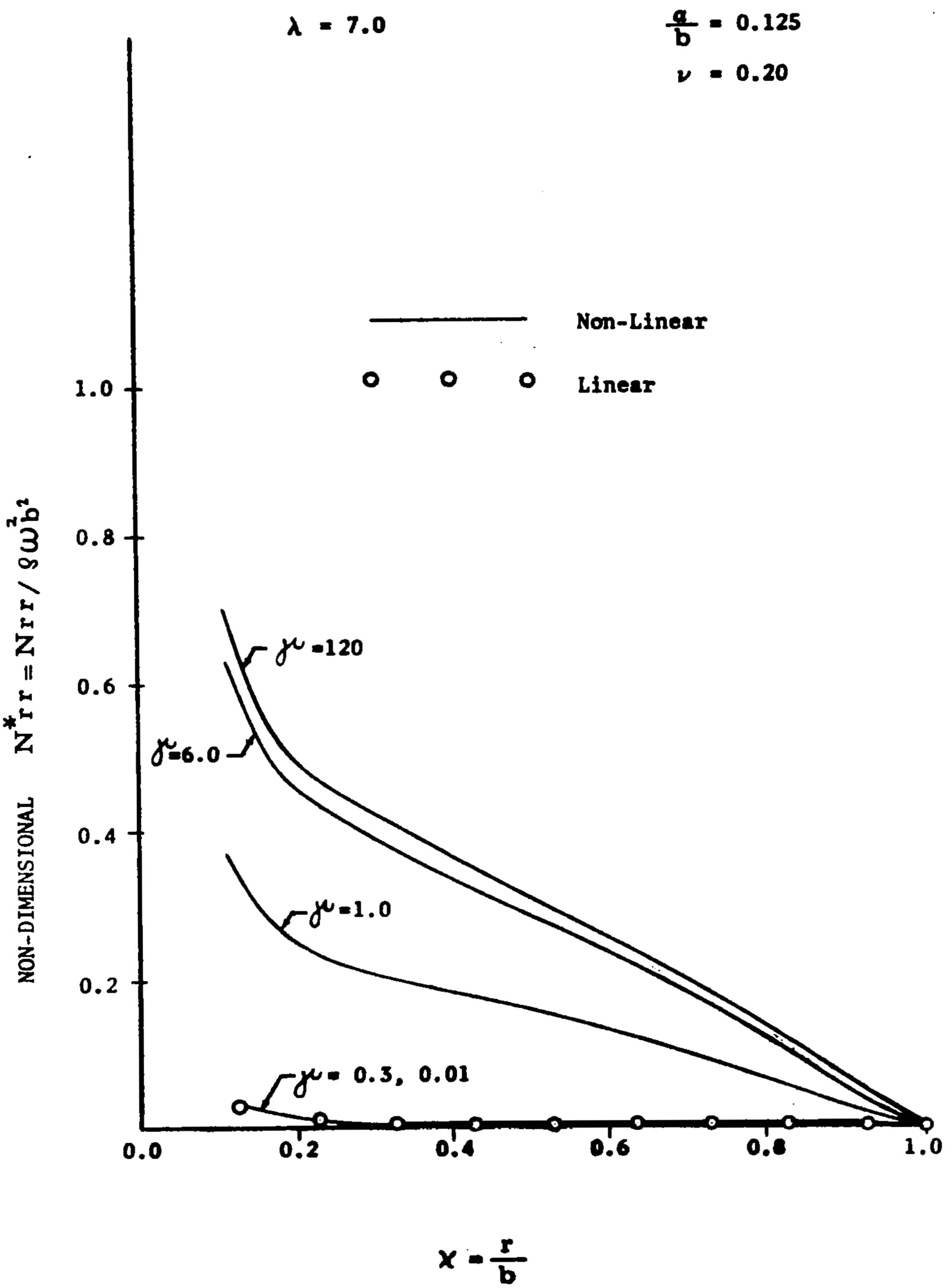


Figure 28. Radial Direct Stress Resultant for Spinning Shell with Fully Clamped Hub. Linear and Nonlinear Theory with λ =7.0 and ν =0.20.

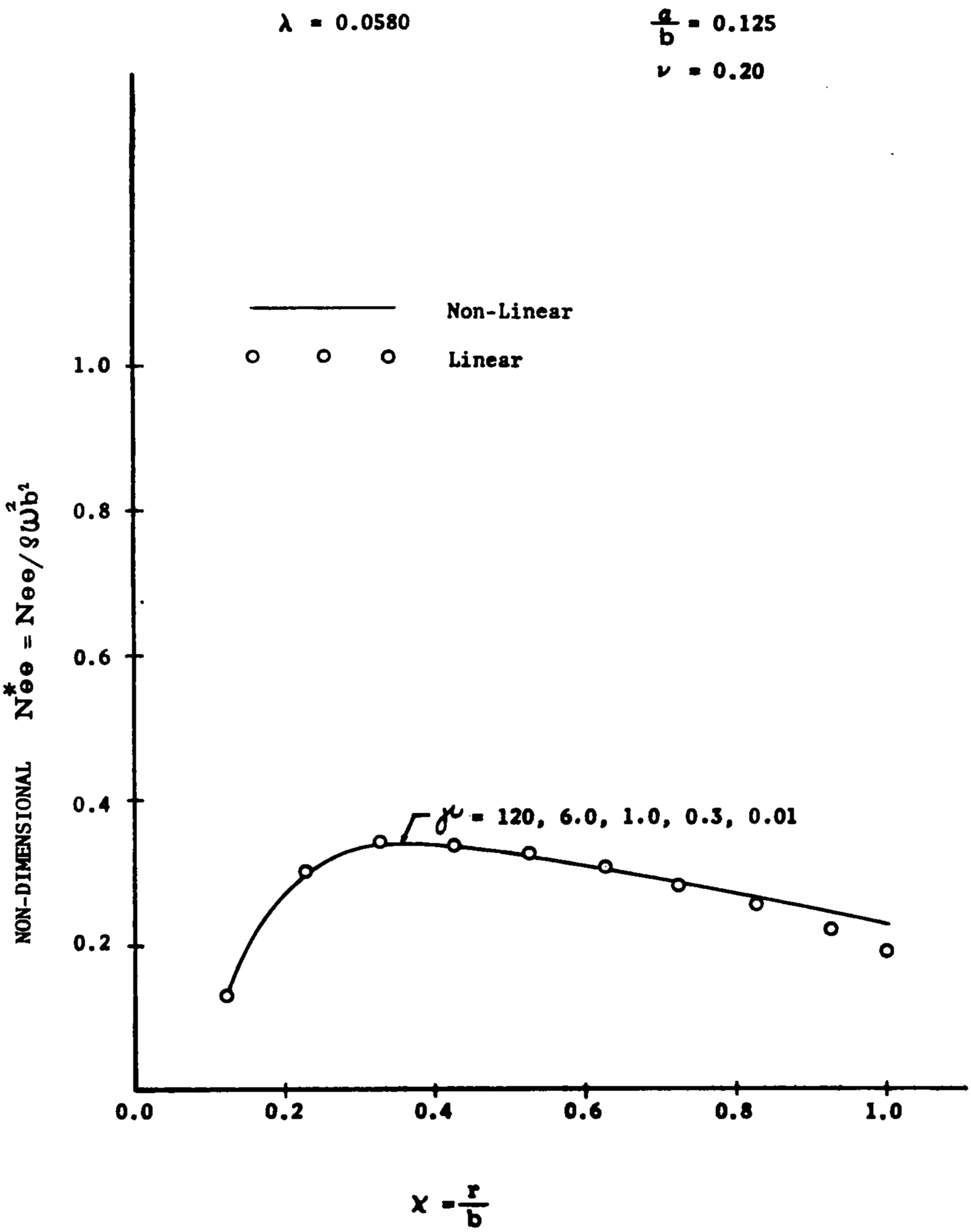


Figure 29. Tangential Direct Stress Resultant for Spinning Shell with Fully Clamped Hub. Linear and Nonlinear Theory with λ =0.058 and ν =0.20.

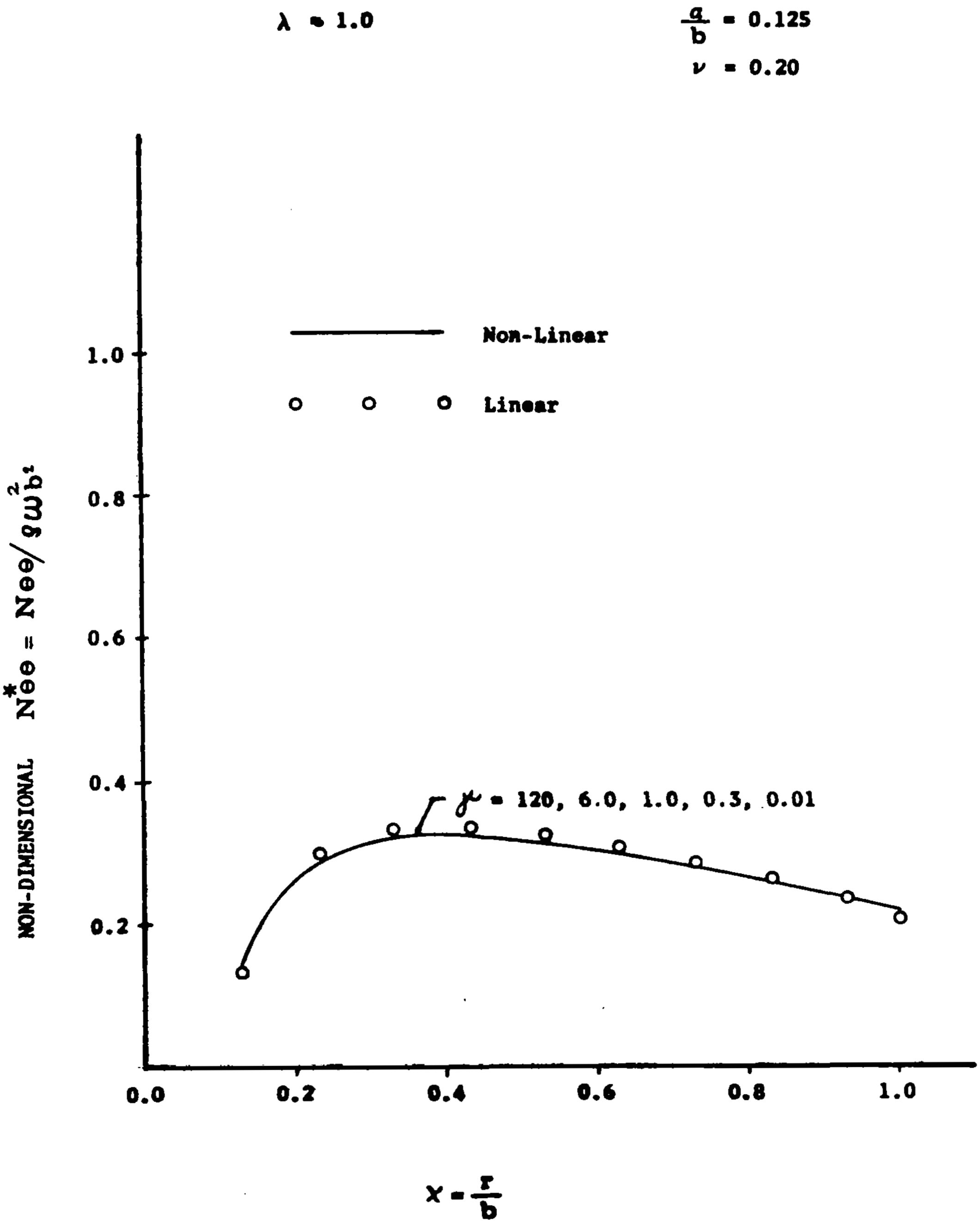


Figure 30. Tangential Direct Stress Resultant for Spinning Shell with Fully Clamped Hub. Linear and Nonlinear Theory with λ =1.0 and ν =0.20.

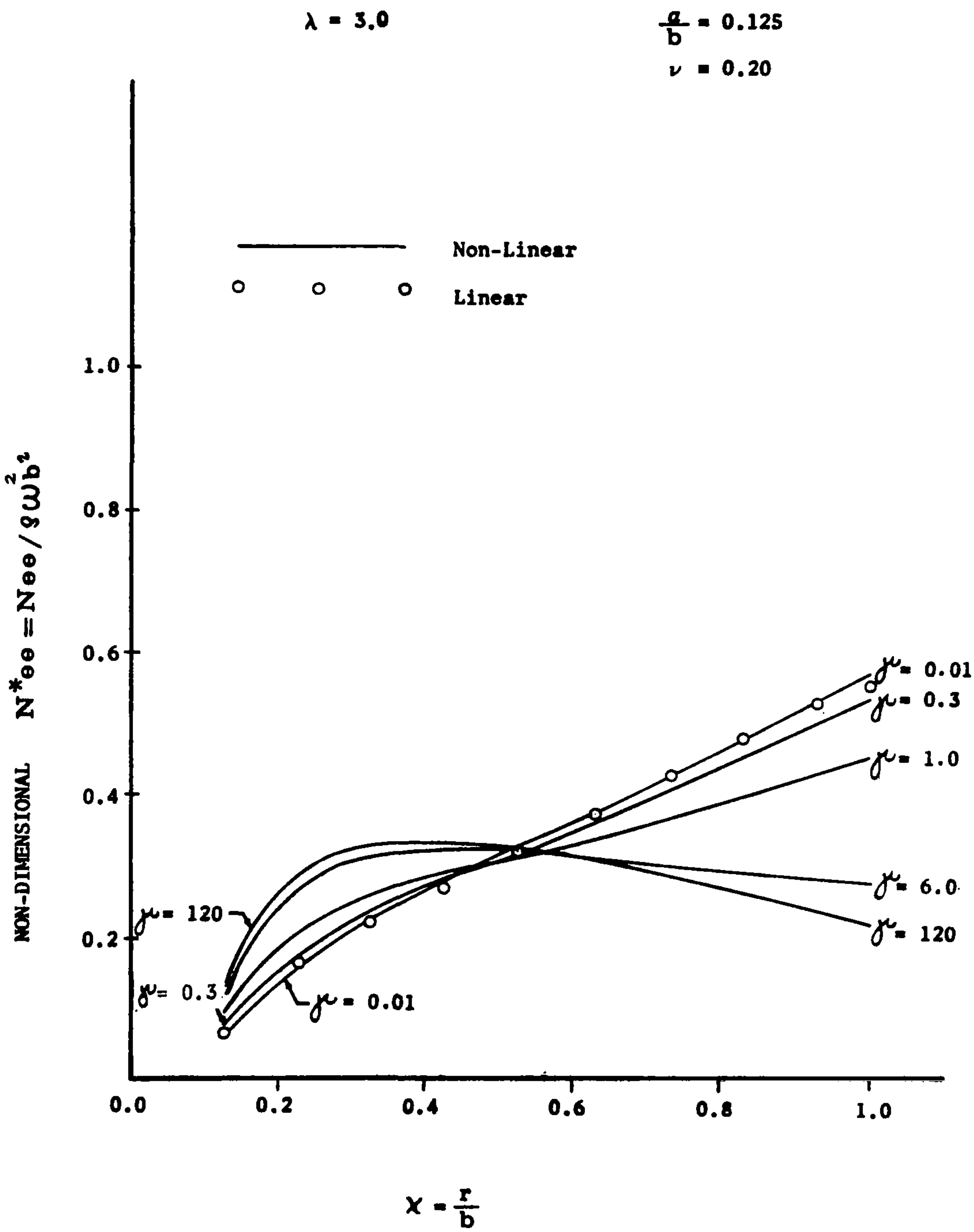


Figure 31. Tangential Direct Stress Resultant for Spinning Shell with Fully Clamped Hub. Linear and Nonlinear Theory with λ =3.0 and ν =0.20.

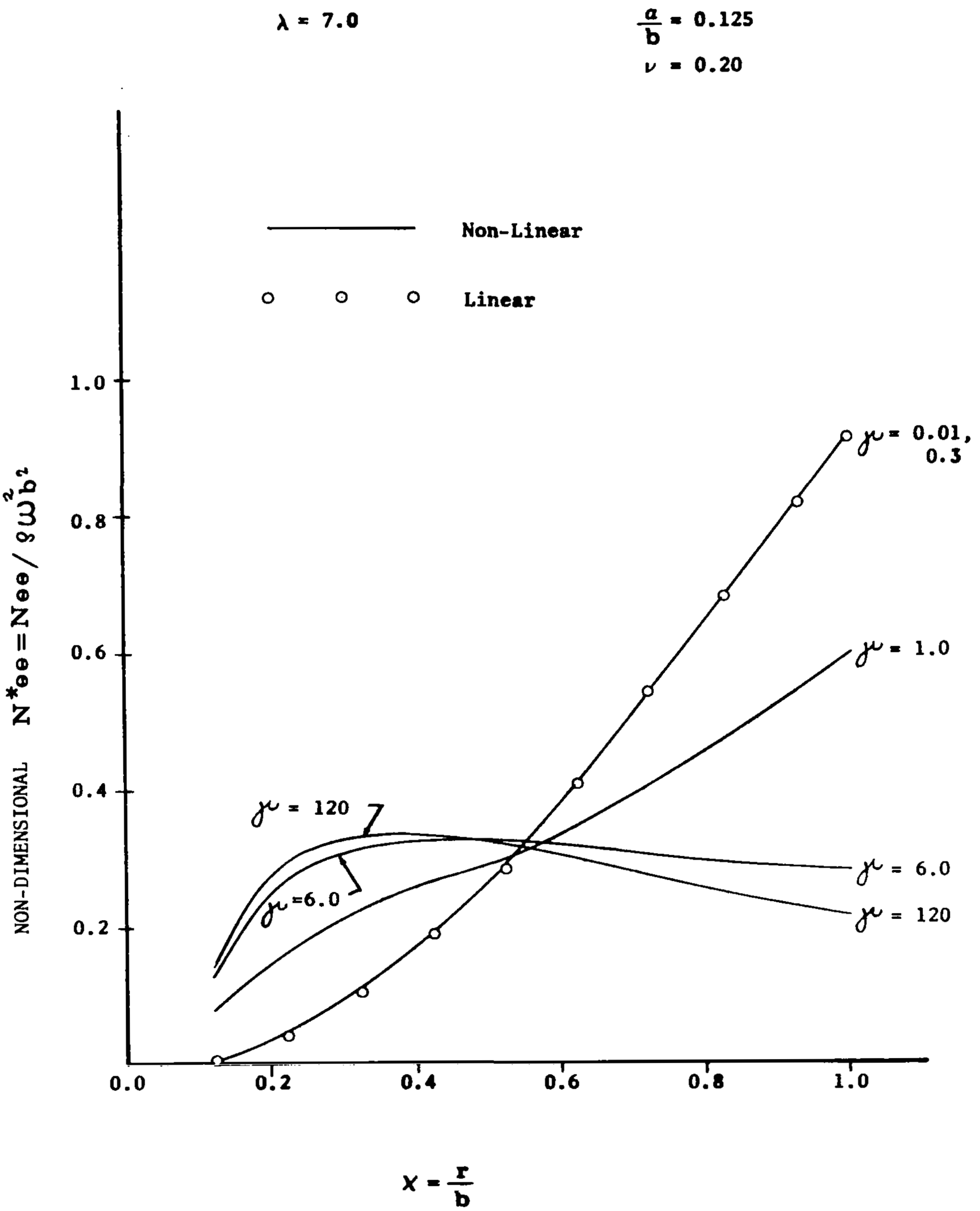


Figure 32. Tangential Direct Stress Resultant for Spinning Shell with Fully Clamped Hub. Linear and Nonlinear Theory with λ =7.0 and ν =0.20.

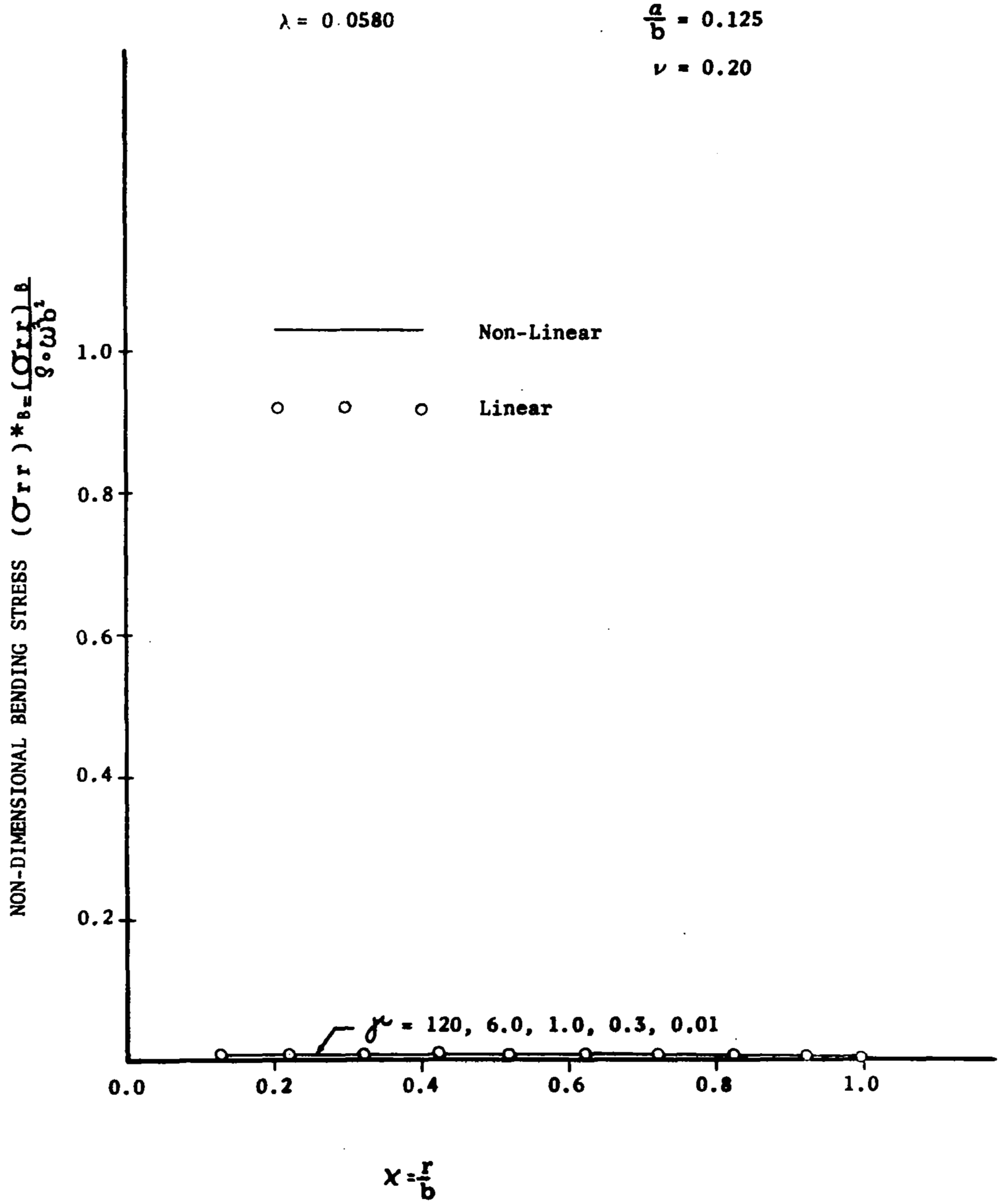


Figure 33. Radial Bending Stress Resultant for Spinning Shell with Fully Clamped Hub. Linear and Nonlinear Theory with λ =0.058 and ν =0.20.

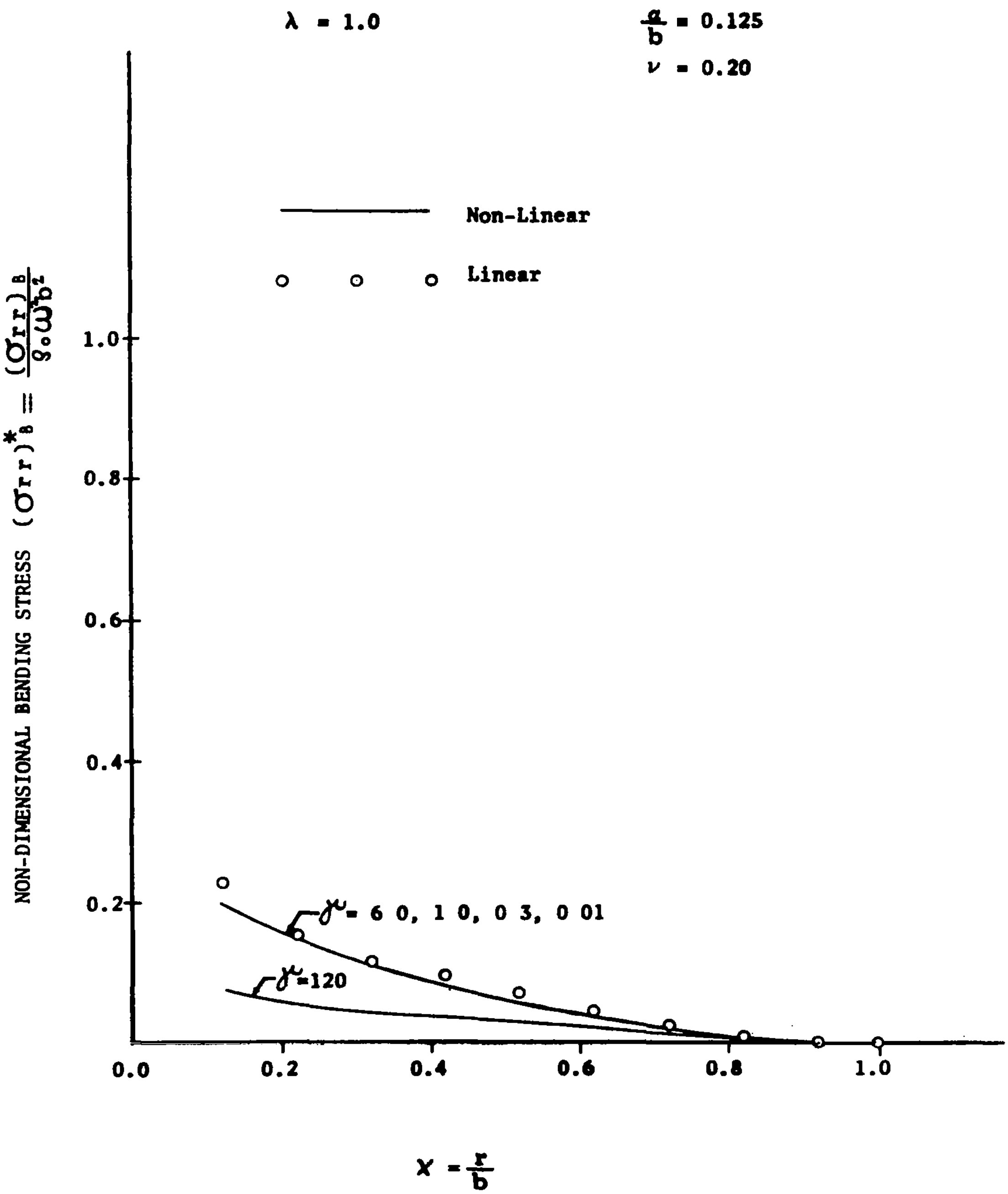


Figure 34. Radial Bending Stress Resultant for Spinning Shell with Fully Clamped Hub. Linear and Nonlinear Theory with λ =1.0 and ν =0.20.

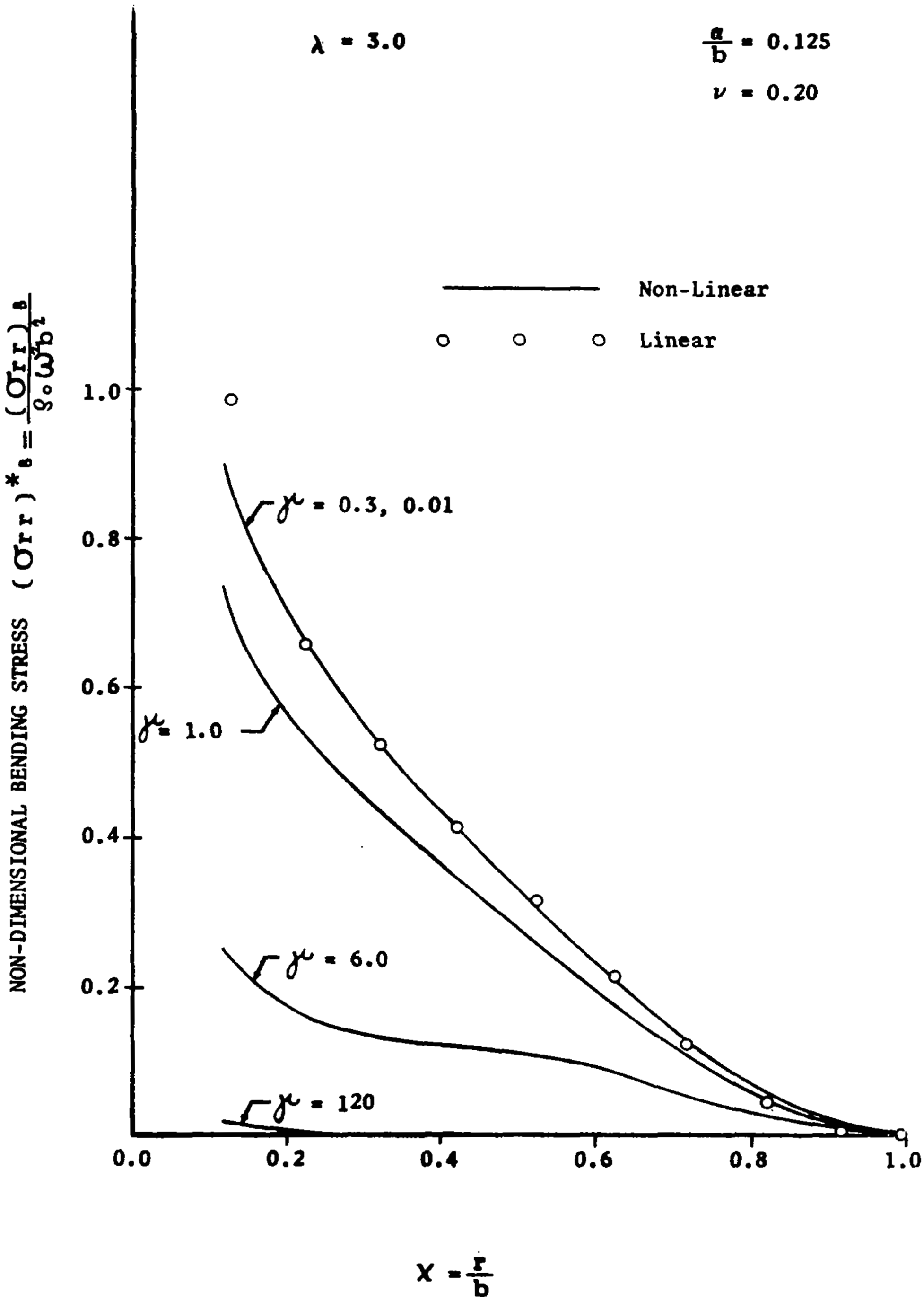


Figure 35. Radial Bending Stress Resultant for Spinning Shell with Fully Clamped Hub. Linear and Nonlinear Theory with λ =3.0 and ν =0.20.

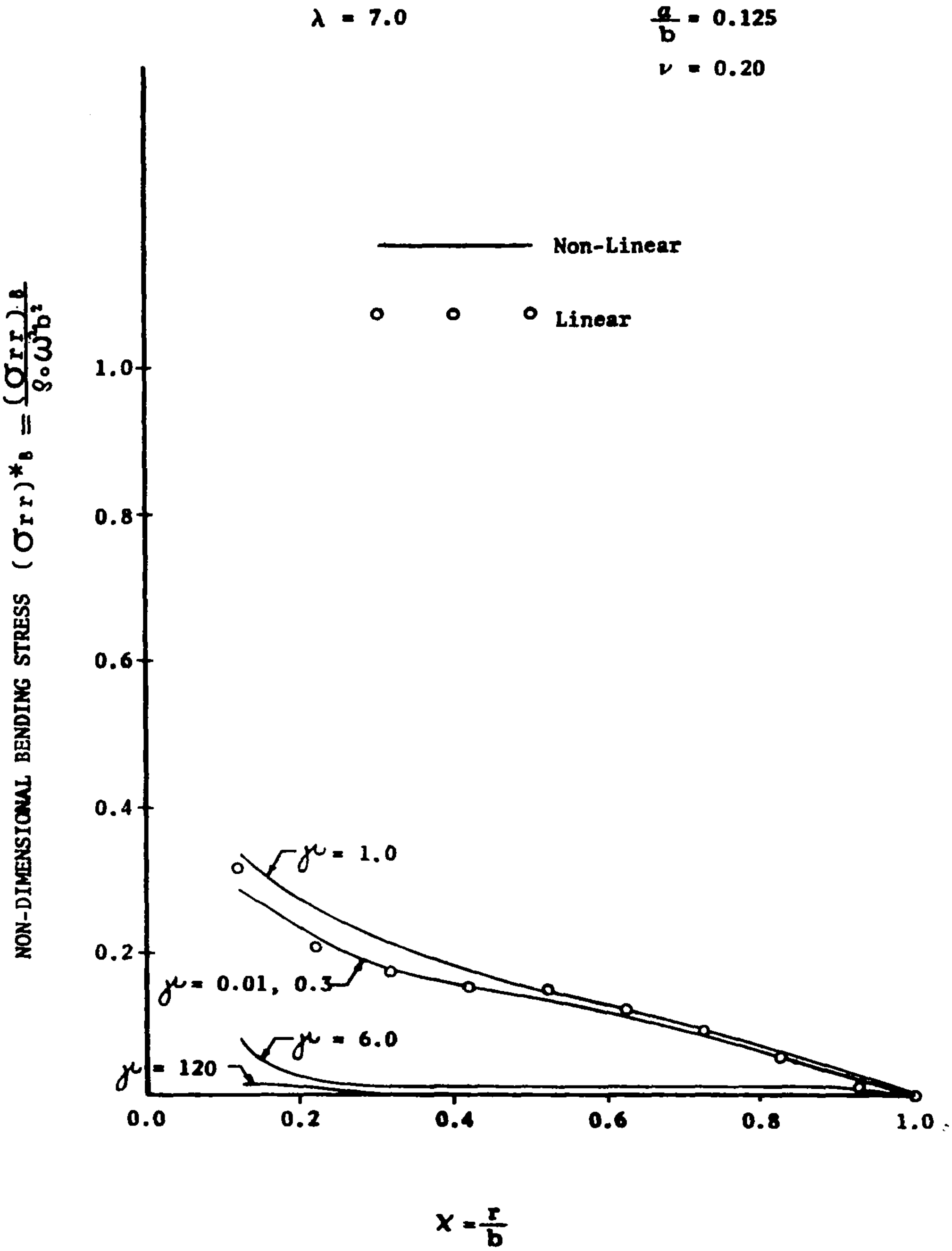


Figure 36. Radial Bending Stress Resultant for Spinning Shell with Fully Clamped Hub. Linear and Nonlinear Theory with λ =7.0 and ν =0.20.

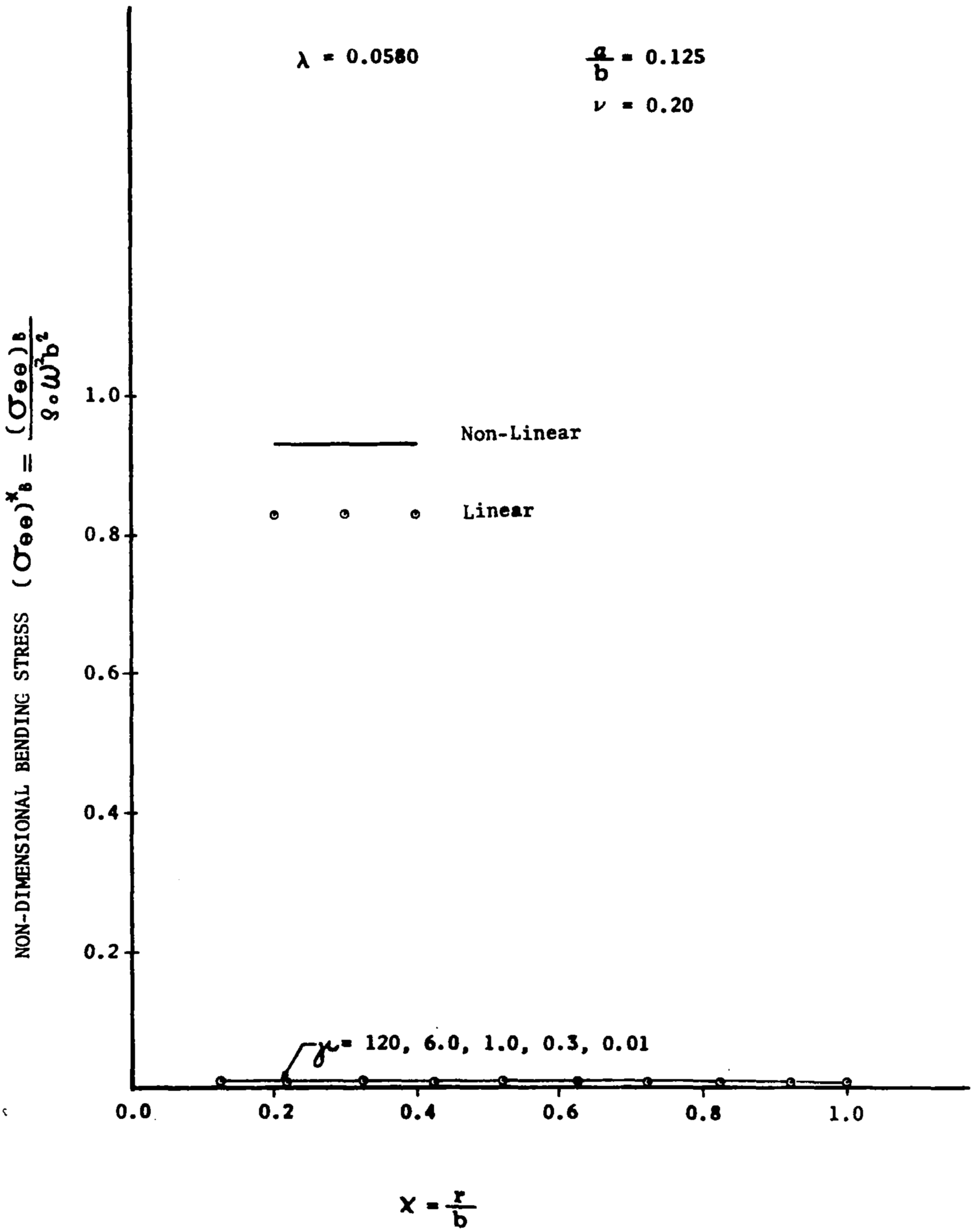


Figure 37. Tangential Bending Stress Resultant for Spinning Shell with Fully Clamped Hub. Linear and Nonlinear Theory with λ =0.058 and ν =0.20.

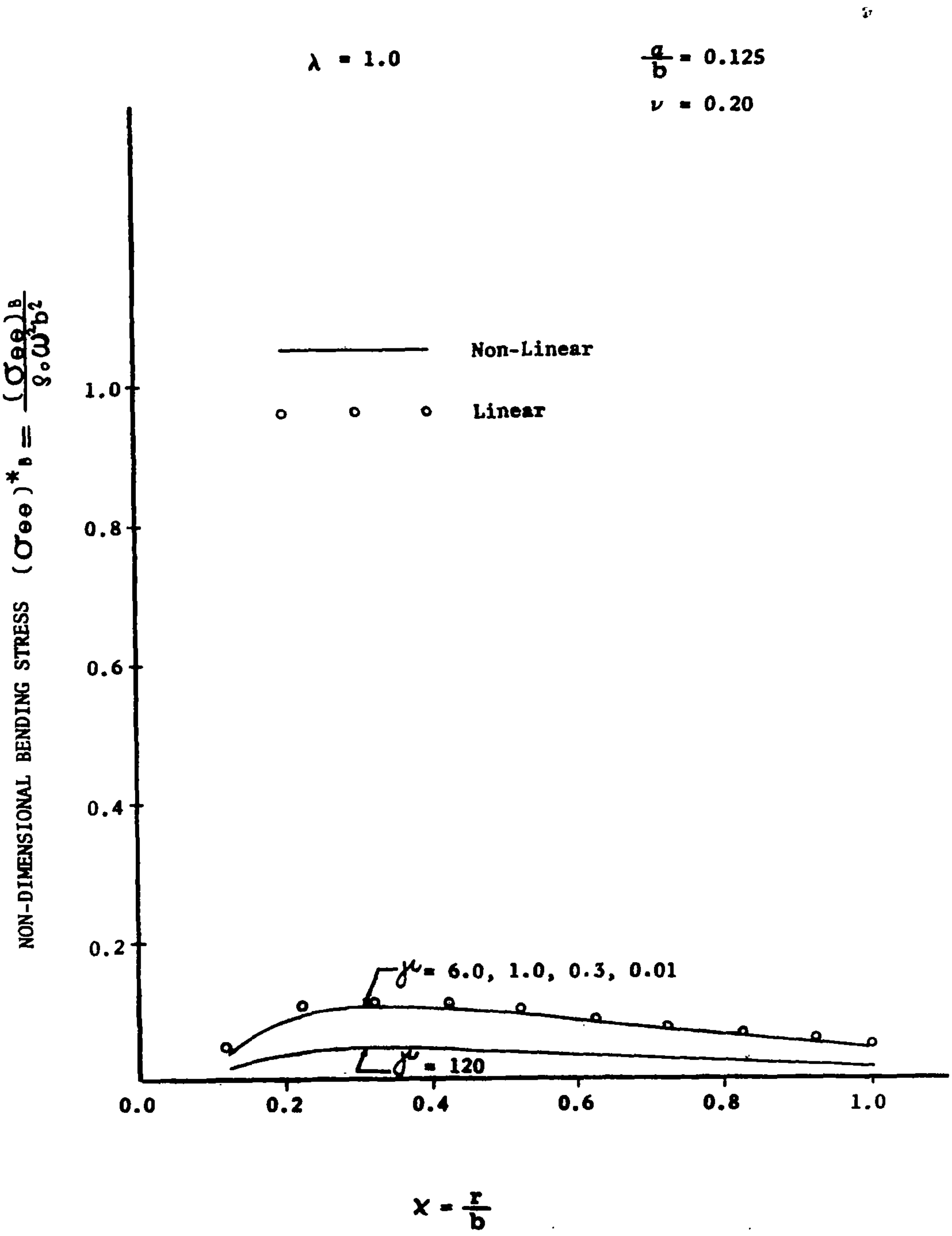


Figure 38. Tangential Bending Stress Resultant for Spinning Shell with Fully Clamped Hub. Linear and Nonlinear Theory with λ =1.0 and ν =0.20.

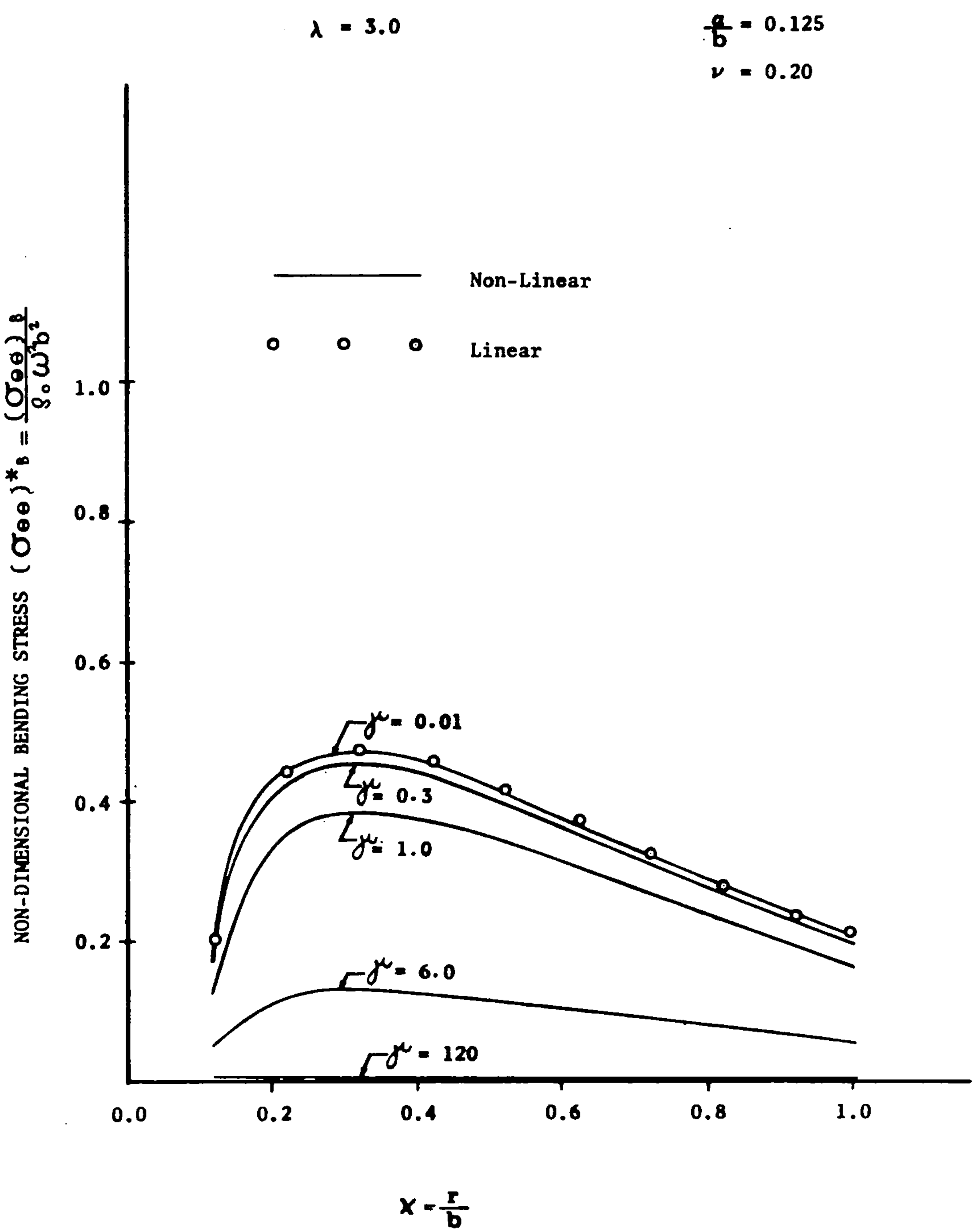


Figure 39. Tangential Bending Stress Resultant for Spinning Shell with Fully Clamped Hub. Linear and Nonlinear Theory with λ =3.0 and ν =0.20.

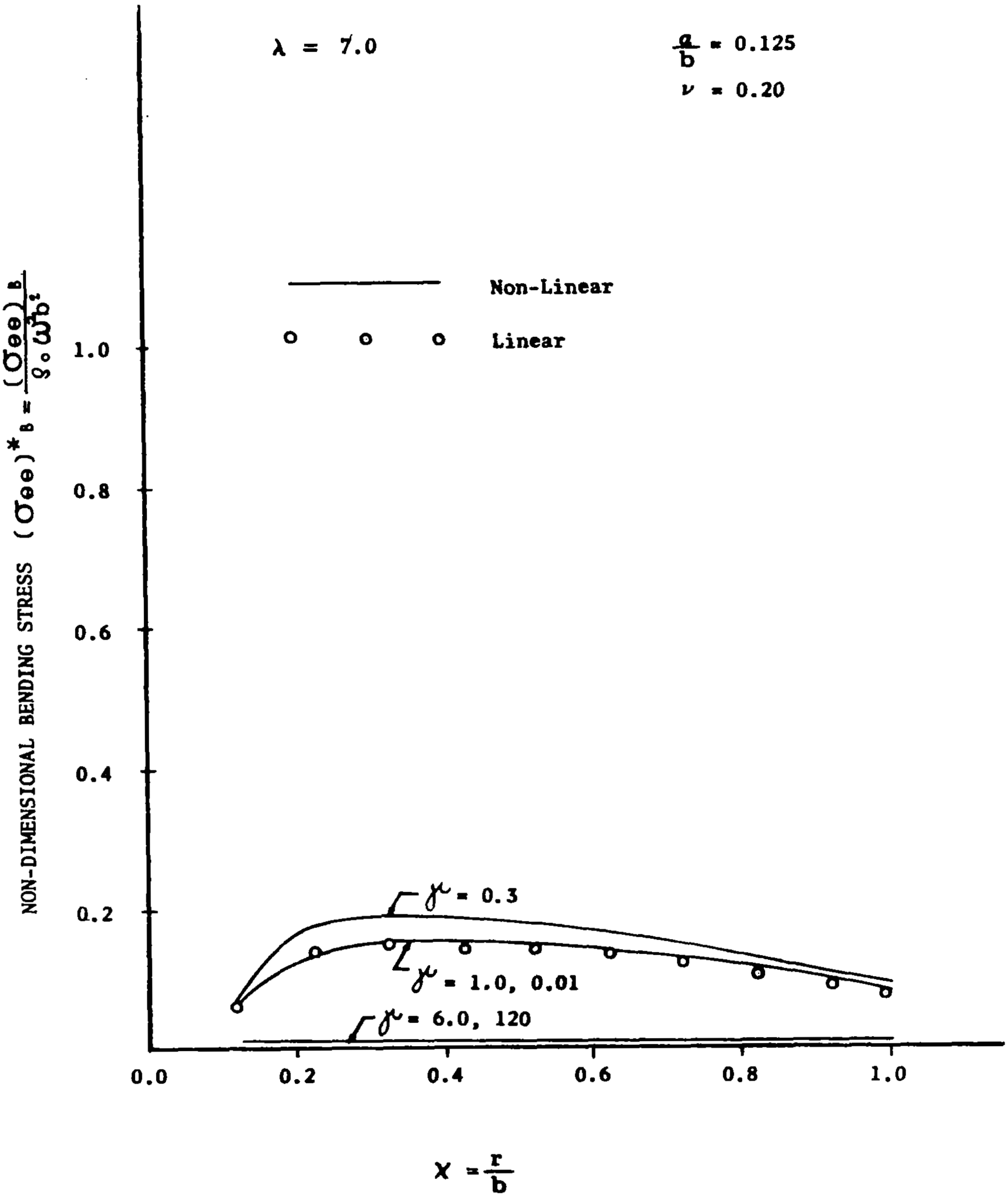


Figure 40. Tangential Bending Stress Resultant for Spinning Shell with Fully Clamped Hub. Linear and Nonlinear Theory with λ =7.0 and ν =0.20.

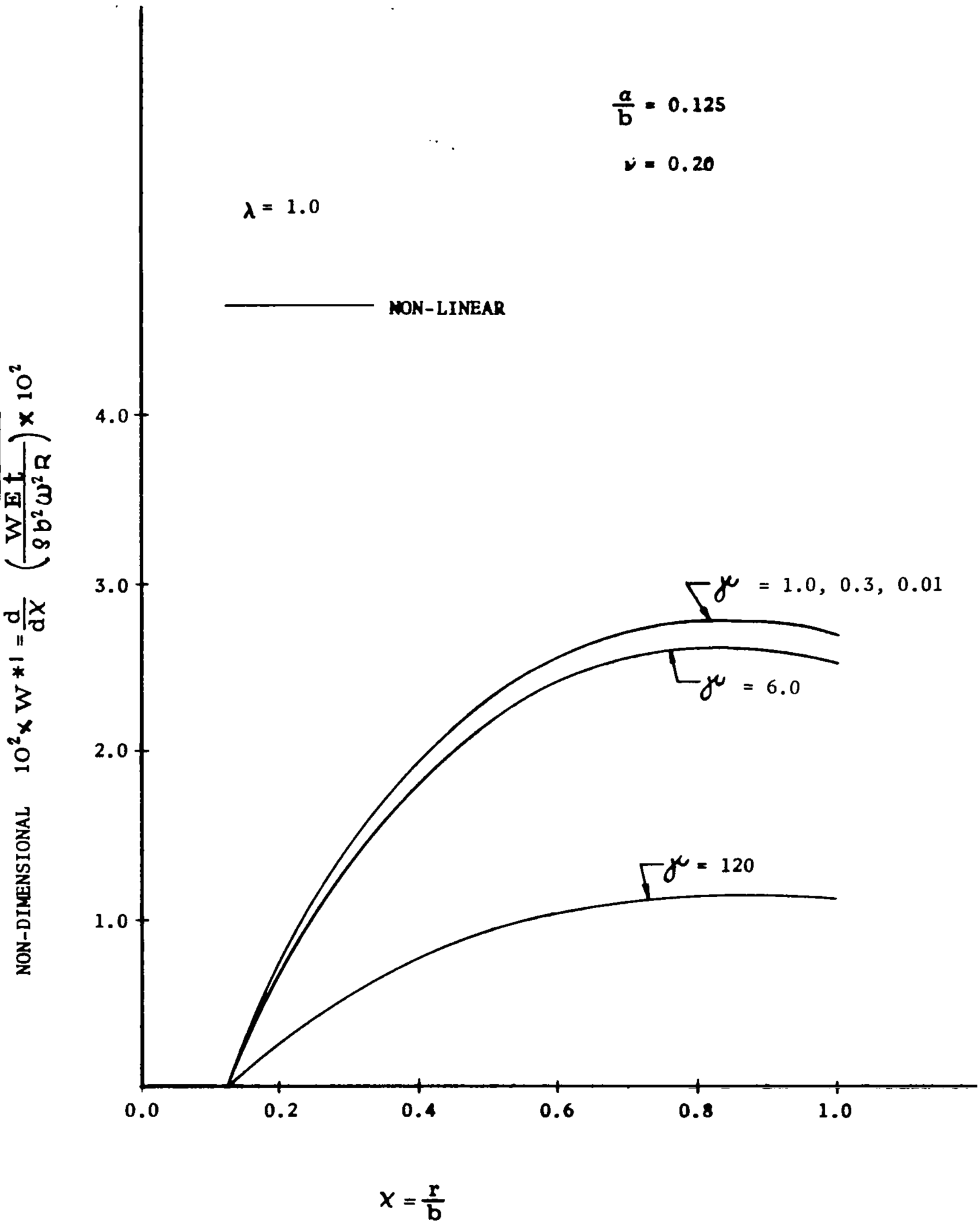


Figure 41. Slope of Deflection Curve for Spinning Shell with Fully Clamped Hub. Nonlinear Theory with λ =1.0 and ν =0.20.

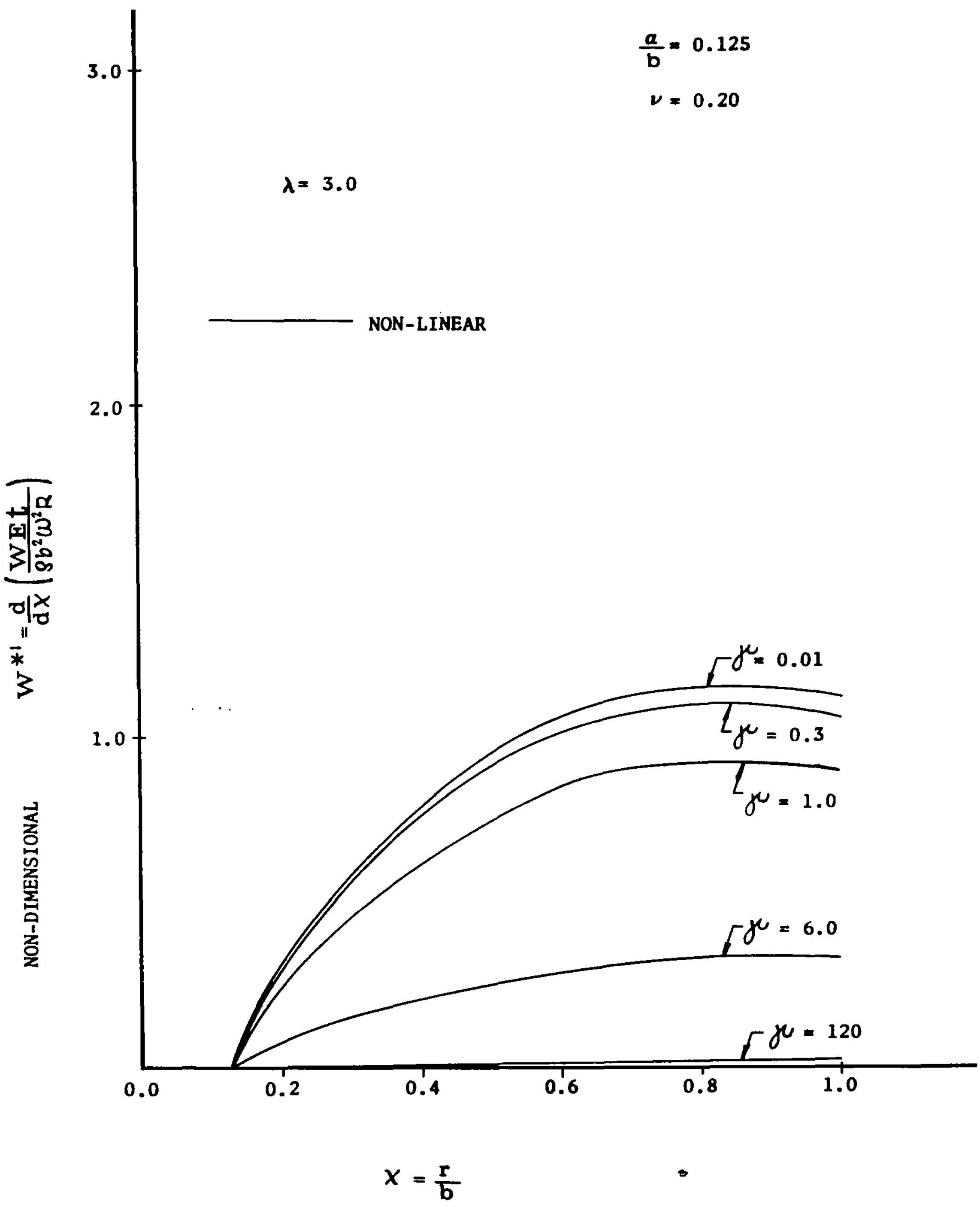


Figure 42. Slope of Deflection Curve for Spinning Shell with Fully Clamped Hub. Nonlinear Theory with λ =3.0 and ν =0.20.

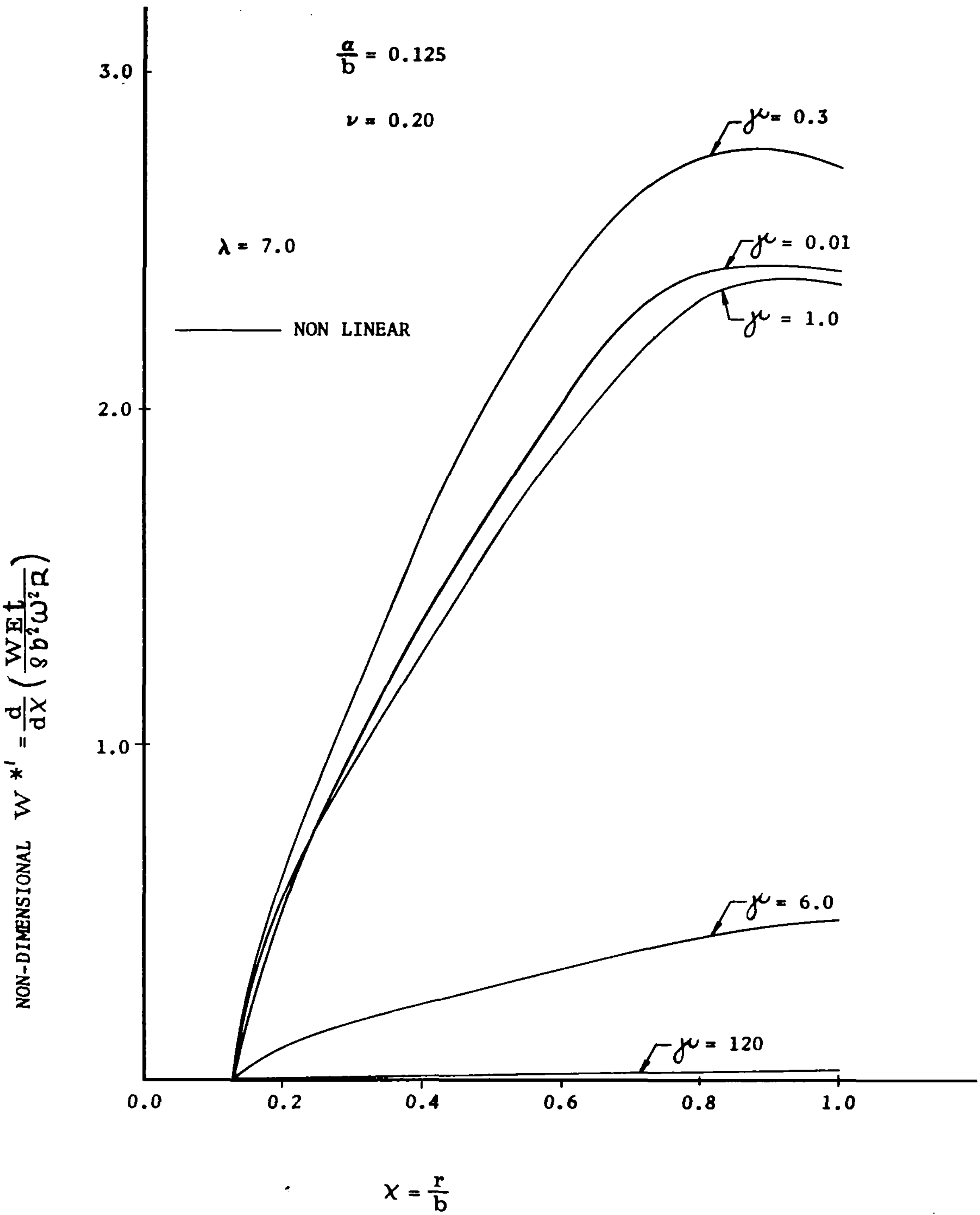


Figure 43. Slope of Deflection Curve for Spinning Shell with Fully Clamped Hub. Nonlinear Theory with λ =7.0 and ν =0.20.

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—NATIONAL AERONAUTICS AND SPACE ACT OF 1958

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